ON THE PIATETSKI-SHAPIRO CONSTRUCTION FOR INTEGRAL MODELS OF SHIMURA VARIETIES

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Abstract. We study the Piatetski-Shapiro construction, which takes a totally real field F and a Shimura datum (G, X) and produces a new Shimura datum (H, Y). If F is Galois, then the Galois group Γ of F acts on (H, Y), and we show that the Γ-fixed points of the Shimura varieties for (H, Y) recover the Shimura varieties for (G, X) under some hypotheses. For Shimura varieties of Hodge type with parahoric level, we show that the same is true for the p-adic integral models constructed by Pappas–Rapoport, if p is unramified in F. We also study the Γ-fixed points of the Igusa stacks of [4] for (H, Y) and prove optimal results.

Contents

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1. Introduction

Let (G,X) be a Shimura datum of Hodge type and let F be a Galois totally real field with Galois group $\Gamma = \mathrm{Gal}(\mathsf{F}/\mathbb{Q})$. Define $\mathsf{H} := \mathrm{Res}_{\mathsf{F}/\mathbb{Q}} G_F$ and let Y be the Shimura datum for H induced by X; we call (H,Y) the *Piatetski-Shapiro construction* for (G,X) associated to F. This is *not* of Hodge type if $[\mathsf{F}:\mathbb{Q}] > 1$, but there is a modified Shimura datum $(\mathsf{H}_1,\mathsf{Y}_1) \hookrightarrow (\mathsf{H},\mathsf{Y})$ that is of Hodge type, see Section 3.6. Then Γ acts on $(\mathsf{H}_1,\mathsf{Y}_1)$ and there is a morphism $(\mathsf{G},\mathsf{X}) \to (\mathsf{H}_1,\mathsf{Y}_1)$ that is Γ-equivariant. Thus if $K \subset \mathsf{H}_1(\mathbb{A}_f)$ is a Γ-stable compact open subgroup, then there is a morphism of Shimura varieties $\mathbf{Sh}_{K^\Gamma}(\mathsf{G},\mathsf{X}) \to \mathbf{Sh}_K(\mathsf{H}_1,\mathsf{Y}_1)$, which induces a map (where the superscript Γ denotes taking Γ-fixed points)

$$\mathbf{Sh}_{K^\Gamma}(\mathsf{G},\mathsf{X}) \to \mathbf{Sh}_K(\mathsf{H}_1,\mathsf{Y}_1)^\Gamma.$$

This morphism can be arranged to be an isomorphism under minor hypotheses, see Theorem 3.6.4. In this paper we will investigate to what extend this result extends to p-adic integral models, and prove affirmative results if p is unramified in F. Our main results will be used in joint work in progress of the authors [11] to construct new exotic Hecke correspondences between the special fibers of different Shimura varieties.

1.1. **Main results.** Let (G,X) be a Shimura datum of Hodge type with reflex with E. Fix a prime $p > 2$ and a prime v of E above p, let E be the completion of E at v and let $\mathcal{O}_E$ be its ring of integers. Let $G = \mathsf{G} \otimes \mathbb{Q}_p$ and let $\mathcal{G}$ be a parahoric model of $\mathcal{G}$ over $\mathbb{Z}_p$.

Let F and Γ be as above, let $F = \mathsf{F} \otimes \mathbb{Q}_p$ and let $\mathcal{O}_F$ be the integral closure of $\mathbb{Z}_p$ in F. Set $\mathcal{H} = \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}_p} \mathcal{G}_{\mathcal{O}_F}$, let $K_p = \mathcal{H}(\mathbb{Z}_p)$ and $K_{1,p} = K_p \cap H_1(\mathbb{Q}_p)$. For a Γ-stable and neat compact open subgroup $K^p \subset \mathsf{H}(\mathbb{A}_f^p)$ we write $K_1^p = K_1^p \cap H_1(\mathbb{A}_f^p)$ and $K_1 = K_1^p K_{1,p}$. We will write $\mathscr{S}_{K_1}(\mathsf{H},\mathsf{Y})$ for the integral model over $\mathcal{O}_E$ of $\mathbf{Sh}_K(\mathsf{H}_1,\mathsf{Y}_1)$ constructed by [5, Theorem I], cf. [27, Theorem 4.5.2]; this has a natural Γ-action by [5, Corollary 4.0.11].

Theorem 1. *Theorem 5.1.2] Assume that F is tamely ramified over $\mathbb{Q}$, that $Ш^1(\mathbb{Q},\mathsf{G}) \to Ш^1(\mathsf{F},G)$ is injective, that $p > 2$ and that p is unramified in F.* ***(1):*** *There is a cofinal* *collection of Γ-stable compact open subgroups $K^p \subset \mathsf{H}(\mathbb{A}_f^p)$ such that the natural map*

$$\mathscr{S}_{K^\Gamma}(\mathsf{G},\mathsf{X}) \to \mathscr{S}_{K_1}(\mathsf{H}_1,\mathsf{Y}_1)^\Gamma \tag{1.1.1}$$

*is a universal homeomorphism. **(2):** If p is coprime to $|\Gamma|\cdot|\pi_1(G^{\mathrm{der}})|$, and if G splits over a tamely ramified extension, then there is a cofinal collection of K^p as above such that the natural map* (1.1.1) *is an isomorphism. **3)** If K_p is hyperspecial, then there is a cofinal collection of K^p as above such that the natural map* (1.1.1) *is an isomorphism.*

The assumption that $Ш^1(\mathbb{Q},\mathsf{G}) \to Ш^1(\mathsf{F},G)$ is injective is necessary, see Section 3.1 for the case of tori. It should be possible to remove it if one works with the extended Shimura varieties of Xiao–Zhu, or equivalently the rational Shimura varieties of Sempliner–Taylor, see [34]. The assumption that p is unramified is also necessary. The assumption that G splits over a tamely ramified extension in part (2) can be weakened, see Hypothesis 4.3.2.

Example 1.1.1. If $(\mathsf{G},\mathsf{X}) = (\mathrm{GL}_2,\mathsf{H})$, then $(\mathsf{H}_1,\mathsf{Y}_1)$ is the subgroup of $\mathrm{Res}_{\mathsf{F}/\mathbb{Q}}\mathrm{GL}_2$ consisting of those matrices with determinant in $\mathbb{G}_m \subset \mathrm{Res}_{\mathsf{F}/\mathbb{Q}}\mathbb{G}_m$. The p-adic integral models for the Shimura varieties for $(\mathsf{H}_1,\mathsf{Y}_1)$ have a moduli interpretation in terms of (weakly polarized) abelian varieties A of dimension $[\mathsf{F}:\mathbb{Q}]$ up to prime-to-p isogeny, equipped with an action $i: \mathcal{O}_{F,(p)} \to \mathrm{End}(A)$. The action of an element $\gamma \in \Gamma$ is then by precomposing i with $\gamma: \mathcal{O}_{F,(p)} \to \mathcal{O}_{F,(p)}$. The morphism from the modular curve can be thought of as taking an elliptic curve up to prime-to-p isogeny E, and forming the abelian variety up to isogeny $E \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{F,(p)}$ together with its tautological $\mathcal{O}_{\mathsf{F},(p)}$ action. The statement of Theorem 1, up to keeping track of level structures, is then essentially an instance of étale descent of quasi-coherent sheaves for the étale cover $\mathrm{Spec}\,\mathcal{O}_{\mathsf{F},(p)} \to \mathrm{Spec}\,\mathbb{Z}_{(p)}$.

Remark 1.1.2. The group $K_{p,1} \subset H_1(\mathbb{Q}_p)$ is generally not a parahoric subgroup, but only a quasi-parahoric subgroup; this causes some technical issues in part (1). To solve them, we appeal to the forthcoming [5], which proves that integral models with quasi-parahoric level have good properties.

1.1.3. *Igusa stacks.* Igusa stacks are certain p-adic analytic objects (Artin v-stacks) associated to a Shimura datum (G,X); they were conjectured to exist by Scholze. They were recently constructed by Zhang [39] in the PEL type case, and in [4] in the Hodge type case.

Let (G,X) be a Shimura datum of Hodge type and a place v above p of the reflex field E of (G,X). Let F,Γ and $(\mathsf{H}_1,\mathsf{Y}_1)$ be as above. Then there is a v-sheaf $\mathrm{Igs}\,(\mathsf{H}_1,\mathsf{Y}_1)$ equipped with a Γ-equivariant map $\mathrm{Igs}\,(\mathsf{H}_1,\mathsf{Y}_1) \to \mathrm{Bun}_{H_1}$ to the stack Bun_{H_1} of H_1-bundles on the Fargues–Fontaine curve, see [4, Theorem I, Theorem II]. This map moreover factors through the open substack $\mathrm{Bun}_{H_1,\mu^{-1}} \subset \mathrm{Bun}_{H_1}$ corresponding to the set $B(H_1,\mu^{-1}) \subset B(H_1)$ of μ^{-1}-admissible σ-conjugacy classes, where μ is the $H_1(\overline{\mathbb{Q}}_p)$ conjugacy class of cocharacters of H_1 induced by the Hodge cocharacter and the place v. We have similar objects for (G,X).

Theorem 2 (Theorem 6.2.1)**.** *If $Ш^1(\mathbb{Q},\mathsf{G}) \to Ш^1(\mathsf{F},G)$ is injective, then the natural map*

$$\mathrm{Igs}\,(\mathsf{G},\mathsf{X}) \to \mathrm{Igs}\,(\mathsf{H}_1,\mathsf{Y}_1)^\Gamma \times_{\left(\mathrm{Bun}_{H_1,\mu^{-1}}\right)^{h\Gamma}} \mathrm{Bun}_{G,\mu^{-1}}$$

is an isomorphism.

Here the superscript $h\Gamma$ denotes the stacky (or homotopy) fixed points and the superscript Γ denotes the usual fixed points of a sheaf. As a consequence of Theorem 2 we prove Theorem 6.3.1, which is an analogue of Theorem 1 on the level of v-sheaves, without the assumption that p is unramified in F or the assumption that F is tamely ramified over $\mathbb{Q}$; we expect this result to be essentially optimal.

Remark 1.1.4. The reason for taking the fiber product in the statement of Theorem 2 is because

$$\mathrm{Bun}_{G,\mu^{-1}} \to \mathrm{Bun}_{H_1,\mu^{-1}}^{h\Gamma}$$

is typically not an isomorphism.

1.2. **Applications.** Our motivation for proving Theorem 1 is that it can be used to reduce questions about the integral models for (G,X) to Γ-equivariant questions about the integral models for (H,Y) or $(\mathsf{H}_1,\mathsf{Y}_1)$. This is useful, because given (G,X) and a prime p, one can always choose a Galois totally real field F that is unramified at p such that $H \otimes \mathbb{Q}_p$ is quasi-split.

For example, in forthcoming work, Xiao and Zhu construct exotic Hecke correspondences between the mod p fibers of different Shimura varieties of Hodge type at unramified and quasi-split primes. In their work, the Shimura varieties correspond to certain Shimura data (G,X) and $(\mathsf{G}',\mathsf{X}')$ such that G and G' are pure inner forms and such that $\mathsf{G}\otimes\mathbb{A}_f \simeq \mathsf{G}'\otimes\mathbb{A}_f$. They prove these correspondences exist and are nonempty by reducing to the case of tori using the existence of CM lifts; this latter fact heavily uses the fact that G and G' are both quasi-split.

These correspondences conjecturally exist for certain pairs (G,X) and $(\mathsf{G}',\mathsf{X}')$, where G and G' are pure inner forms, under the more general condition that $\mathsf{G} \otimes \mathbb{A}_f^p \simeq \mathsf{G}' \otimes \mathbb{A}_f^p$. In work in preparation of the authors, see [11], we construct these more general exotic Hecke correspondences for primes p where G splits over an unramified extension (but is not necessarily quasi-split). As a consequence, we deduce the existence of exotic isomorphisms between (perfect) Igusa varieties for (G,X) and $(\mathsf{G}',\mathsf{X}')$, generalizing [3].

The rough idea is to choose a Galois totally real field extension F such that $\mathsf{H}_1 \otimes \mathbb{A}_f \simeq \mathsf{H}_1' \otimes \mathbb{A}_f$ and such that H_1 is quasi-split. We then take the Γ-fixed-points of the correspondences between $(\mathsf{H}_1,\mathsf{Y}_1)$ and $(\mathsf{H}_1',\mathsf{Y}_1')$ constructed by Xiao–Zhu and show that the resulting correspondence spaces are nonempty. By Theorem 1, these then give correspondences between the Shimura varieties for (G,X) and $(\mathsf{G}',\mathsf{X}')$.

1.3. **Proofs of the main theorems.** The proof of Theorem 1 proceeds in two steps. We first prove the theorem on the generic fiber, and then extend to integral models.

1.3.1. To prove Theorem 1 over $\mathbb{C}$, we argue on the level of $\mathbb{C}$-points and reduce to a concrete question about fixed points of adelic double quotients. To tackle such questions, we use the methods of non-abelian group cohomology. Results from [35] immediately imply that for $K \subset \mathsf{H}_1(\mathbb{A}_f)$ a Γ-stable compact open subgroup, the natural map

$$\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}_f)/K^\Gamma \to (\mathsf{H}_1(\mathbb{Q})\backslash\mathsf{H}_1(\mathbb{A}_f)/K)^\Gamma$$

is a bijection if both $\mathrm{Ker}(H^1(\Gamma, \mathsf{H}_1(\mathbb{Q})) \to H^1(\Gamma, \mathsf{H}_1(\mathbb{A}_f)))$ and $H^1(\Gamma, K)$ are trivial. The first kernel seems hard to understand directly, and it is not clear to us how to show that there are any Γ-stable compact open subgroups $K \subset \mathsf{H}_1(\mathbb{A}_f)$ with trivial Γ-cohomology.

The situation is much better when we work with H instead of H_1. The kernel

$$\mathrm{Ker}(H^1(\Gamma, \mathsf{H}(\mathbb{Q})) \to H^1(\Gamma, \mathsf{H}(\mathbb{A}_f)))$$

can be identified with the kernel of $Ш^1(\mathbb{Q}, \mathsf{G}) \to Ш^1(\mathsf{F}, G)$, and if F is tamely ramified over $\mathbb{Q}$ then we prove that there is a cofinal collection of Γ-stable $K \subset \mathsf{H}(\mathbb{A}_f)$ with $H^1(\Gamma, K)$ trivial, see Section 3.4. The general theory tells us that

$$\left[\mathsf{G}(\mathbb{Q})\backslash \mathsf{X} \times \mathsf{G}(\mathbb{A}_f)/K^\Gamma\right] \to [\mathsf{H}(\mathbb{Q})\backslash \mathsf{Y} \times \mathsf{H}(\mathbb{A}_f)/K]^{h\Gamma}$$

is an equivalence if $Ш^1(\mathbb{Q}, \mathsf{G}) \to Ш^1(\mathsf{F}, G)$ is injective and $H^1(\Gamma, K) = \{1\}$, see Theorem 3.5.1. We give a proof of this statement relying on some 2-category theory; we give a self-contained exposition of the results that we need in Appendix A.

To use this to prove Theorem 1 over $\mathbb{C}$, we argue as follows: We know that the composition

$$\mathsf{G}(\mathbb{Q})\backslash \mathsf{X} \times \mathsf{G}(\mathbb{A}_f)/K^\Gamma \to (\mathsf{H}_1(\mathbb{Q})\backslash \mathsf{Y}_1 \times \mathsf{H}_1(\mathbb{A}_f)/K_1)^\Gamma \to [\mathsf{H}(\mathbb{Q})\backslash \mathsf{Y} \times \mathsf{H}(\mathbb{A}_f)/K]^{h\Gamma}$$

is an equivalence of groupoids, by the results discussed in the previous paragraph. It thus suffices to prove that the second map is fully faithful, which we do under a minor assumption on K, see Proposition 3.6.3. This method forces us to work with the Shimura varieties for $(\mathsf{H}_1, \mathsf{Y}_1)$ of level $K_1 = K \cap \mathsf{H}_1(\mathbb{A}_f)$.

1.3.2. We now explain how to extend Theorem 1 from the generic fiber to integral models. We observe that it suffices to prove that $\mathscr{S}_{K_1}(\mathsf{H}_1, \mathsf{Y}_1)^\Gamma$ is normal and flat over $\operatorname{Spec} \mathcal{O}_E$. If the order of Γ is prime-to-p, then we will prove this by taking Γ-fixed points of the local model diagram for $\mathscr{S}_{K_1}(\mathsf{H}_1, \mathsf{Y}_1)$ constructed by [15], and using the fact that fixed points of smooth morphisms are again smooth. The problem then reduces to showing that Γ-fixed points of the local models for $(\mathsf{H}_1, \mathsf{Y}_1)$ give the local models for (G, X). We deduce this from unramified base change for local models. Here we crucially use the assumption that p is unramified in F.

When p divides the order of Γ, we can only show that the weak normalization of $\mathscr{S}_{K_1}(\mathsf{H}_1, \mathsf{Y}_1)^\Gamma$ is normal and flat over $\operatorname{Spec} \mathcal{O}_E$. For this, we may argue on the level of the corresponding v-sheaves, where we reduce it to proving that the Γ-fixed points of the integral local Shimura varieties for H^{ad} give the integral local Shimura varieties for G^{ad}. Once again here we crucially use the assumption that p is unramified in F.

1.3.3. In order to facilitate the arguments sketched in Section 1.3.2, it is important for us to verify that many constructions in the theory of integral models of Shimura varieties (shtukas, local models, etc) are functorial in the triple $(\mathsf{G}, \mathsf{X}, \mathcal{G})$ in a 2-categorical sense. For example, we show that the morphisms $\mathscr{S}_K(\mathsf{H}, \mathsf{Y})^{\diamond/} \to \mathrm{Sht}_{\mathcal{H},\mu}$ from the (modified) diamond associated to an integral model of a Shimura variety, see Section 2.3.3, to the stack of $\mathcal{G}$-shtukas of type μ of Pappas–Rapoport [27], can be upgraded to a weak natural transformation of weak functors (or pseudo-functors), see Proposition 4.1.4 and Corollary 4.2.2. This is necessary because this implies that there is an induced map

$$\mathscr{S}_K(\mathsf{H},\mathsf{Y})^{\diamondsuit/,\Gamma} \to (\mathrm{Sht}_{\mathcal{H},\mu})^{h\Gamma}.$$

These induced maps play a crucial role in the proof of Theorem 1. In that proof, we make use of the 2-categorical theory of non-abelian Galois cohomology theory for the strict $(2,1)$-category of stacks on an arbitrary site. We develop this theory from scratch in Appendix A.

1.4. **Outline of the paper.** In Section 2 we discuss preliminaries on perfectoid geometry. We also recall local models and local shtukas and compute the fixed points of local models and integral local Shimura varieties. In Section 3 we prove Theorem 1 on the generic fiber. In Section 4 we discuss integral models of Shimura varieties following Pappas–Rapoport [27], and construct a 2-categorical enhancement of their shtukas. In Section 4.3, we discuss local model diagrams for Shimura varieties of Hodge type following [19], and prove Γ-equivariance. In Section 5 we prove Theorem 1. In Section 6 we introduce Igusa stacks, prove Theorem 2 and deduce Theorem 6.3.1.

In Appendix A we recall some 2-category theory used throughout the paper. The main question we answers is when taking (2-categorical) fixed points commutes with taking (2-categorical) quotients.

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2. Preliminaries

2.1. **2-category theory.** At many points in this article, we will be taking fixed points for the action of a finite group Γ on a stack $\mathcal{X}$. The most reasonable way of doing this seems to be taking the 2-categorical or homotopy fixed points. For example, if F is a finite Galois extension of $\mathbb{Q}$ with Galois group Γ, then Γ acts on the category of F-vector spaces by twisting the F-action, and the Γ-homotopy fixed points can be identified with the category of $\mathbb{Q}$-vector spaces.

We will refer the reader to Appendix A for the definition of a stack (or category) with Γ-action and the notion of a Γ-equivariant morphism of stacks (or categories) with Γ-action, see Definition A.1.3. In particular, we note that it is *not* a property of a morphism to be Γ-equivariant, but rather an extra structure. To emphasize this, we will sometimes call such morphisms Γ-*equivariant in the* 2-*categorical sense.*

2.2. **Non-abelian cohomology.** Given a group Γ acting on a (possibly non-abelian) group H, we denote by $H^1(\Gamma, H)$ the pointed set given by 1-cocycles $\sigma : \Gamma \to H$ up to H-conjugacy. We begin with the following simple non-abelian Shapiro's lemma. This is presumably standard material but we were unable to find a satisfactory reference; note, however, that this result also appears in [35, Section 5.8.(b)] as an exercise left to the reader.

Lemma 2.2.1. *Let Γ be a group and let H be a group with an action of a subgroup Γ' of Γ. Then if $X := Map_{\Gamma'}(\Gamma, H)$ with the natural left action $\gamma_0((h_\gamma)_{\gamma\in\Gamma}) = (h_{\gamma\cdot\gamma_0})_{\gamma\in\Gamma}$ we have a natural isomorphism $\psi : H^1(\Gamma, X) \xrightarrow{\sim} H^1(\Gamma', H)$.*

Proof. In the forward direction, the map $\tilde{\psi}$ on cocycles is given by restriction on the level of the domain and the pushforward map induced by evaluation at the identity from X to H. A cocycle representing an element of the set $H^1(\Gamma, X)$ is a homomorphism $\phi : \Gamma \to X \rtimes \Gamma$ such that $\pi \circ \phi = \mathrm{Id}_\Gamma$ where $\pi : X \rtimes \Gamma \to \Gamma$ is the projection, and X is given the obvious group structure. Two cocycles are equivalent if there exists $x \in X$ such that $x\phi x^{-1} = \phi'$. We view H valued cocycles of Γ' in the analogous way.

Given a cocycle $\bar{\phi} \in Z^1(\Gamma', H)$ we can construct $\phi \in Z^1(\Gamma, X)$ as follows: fix a system of representatives $\{\gamma_i\}_{i\in I}$ for $\Gamma'\backslash\Gamma$, then if $\gamma = \tau \cdot \gamma_i$ for $\tau \in \Gamma'$ we define

$$\phi(\gamma) = (x_\gamma, \gamma),$$

where $x_\gamma(1) = \bar{\phi}(\tau)\cdot h_i^\tau$ where $h_i =: \phi(\gamma_i)(1)$ is chosen arbitrarily except for the representative of the identity coset, on which the value of ϕ is already fixed. The cocycle relation dictates that $x_\gamma(\gamma') = x_{\gamma'}(1)^{-1}x_{\gamma'\cdot\gamma}(1)$ (note that this is compatible with our definition of $x_\gamma(1)$ before). This gives a complete definition of ϕ. This construction gives a right inverse of the map $H^1(\Gamma, X) \to H^1(\Gamma', H)$ which shows that it is surjective.

We observed in the previous paragraph that the values of the cocycle ϕ in $Z^1(\Gamma, X)$ are determined by $x_\gamma(1)$ for all $\gamma \in \Gamma$. In particular, if $x_\gamma(1) = 1$ for all $\gamma \in \Gamma$ then ϕ is the trivial cocycle. Suppose $\tilde{\psi}(\phi) \sim \tilde{\psi}(\phi')$, where $\phi(\gamma) = (x_\gamma, \gamma), \phi'(\gamma) = (x'_\gamma, \gamma)$, by the previous paragraph we may assume that $\tilde{\psi}(\phi) = \tilde{\psi}(\phi')$, indeed given a 1-coboundary for Γ' which gives rise to the equivalence of $\tilde{\psi}(\phi), \tilde{\psi}(\phi')$ we can extend it to a 1-coboundary for Γ with values in X using the process above, such that modifying ϕ by this coboundary we have $\tilde{\psi}(\phi) = \tilde{\psi}(\phi')$. Then we can define $y \in X$ by

$$y(\gamma) = x'_\gamma(1)^{-1}x_\gamma(1)$$

we see that

$$y(\tau \cdot \gamma) = x'_{\tau\cdot\gamma}(1)^{-1}x_{\tau\cdot\gamma}(1) = (x'_\tau(1)x'_\gamma(\tau))^{-1}(x_\tau(1)x_\gamma(\tau)) = \tau(x'_\gamma(1)^{-1} \cdot x_\gamma(1))$$

as x_γ, x'_γ are Γ'-equivariant, and so we see that $y \in X$ as y is Γ'-equivariant as well. Then $(y\phi y^{-1})(\gamma) = (z_\gamma, \gamma)$ where $z_\gamma(1) = y(1)x_\gamma(1)y(\gamma)^{-1} = x'_\gamma(1)$ whence y is a coboundary which evidences the equivalence of ϕ and ϕ'. Thus the map ψ is also injective. ☐

2.3. **Background on perfectoid geometry.** In this section we recall some background on perfectoid spaces. We refer the reader to [27] and [33] for details.

Let k be a perfect field of characteristic p and write Perf_k denote the category of perfectoid spaces over k. If $k = \mathbb{F}_p$ we write $\mathrm{Perf} = \mathrm{Perf}_{\mathbb{F}_p}$. For any perfectoid space S over k, we write $S \dot{\times} \mathbb{Z}_p$ for the analytic adic space defined in [33, Proposition 11.2.1]. In particular, when $S = \mathrm{Spa}(R, R^+)$ is affinoid perfectoid, $S \dot{\times} \mathbb{Z}_p$ is given by

$$S \dot{\times} \mathbb{Z}_p = \mathrm{Spa}(W(R^+)) \setminus \{[\varpi] = 0\},$$

where $W(R^+)$ denotes the ring of p-typical Witt vectors of the perfect ring R^+, and where $[\varpi]$ denotes the Teichmülller lift to $W(R^+)$ of a pseudo-uniformiser ϖ in R^+. The Frobenius for $W(R^+)$ restricts to give a Frobenius operator Frob_S on $S \dot{\times} \mathbb{Z}_p$. By [33, Proposition 11.3.1], any untilt $S^\sharp$ of S determines a closed Cartier divisor $S^\sharp \hookrightarrow S \dot{\times} \mathrm{Spa}\,\mathbb{Z}_p$. For S in Perf we define also $\mathcal{Y}(S) = S \dot{\times} \mathbb{Z}_p \setminus \{p = 0\}$.

For any S in Perf, the relative adic Fargues–Fontaine curve over S is the quotient

$$X_S = \mathcal{Y}(S)/\varphi^{\mathbb{Z}}.$$

By [9, Proposition II.1.16], the action of Frob_S on $\mathcal{Y}(R, R^+)$ is free and totally discontinuous, which means that the quotient is well-defined. Let G be a reductive group over $\mathbb{Q}_p$. Following [9], we denote by $\mathrm{Bun}_G(S)$ the groupoid of G-torsors on X_S. By [9, Proposition III.1.3], Bun_G is a small v-stack on Perf. A morphism $f : G \to G'$ induces a 1-morphism $\mathrm{Bun}_G \to \mathrm{Bun}_{G'}$ by pushing out torsors, and the following lemma follows from Lemma A.2.1.

Lemma 2.3.1. *The construction of the v-stack* Bun_G *for* G *a reductive group scheme over* $\mathbb{Q}_p$ *extends to a weak functor*

$$\{\textit{reductive groups over } \mathbb{Q}_p\} \to \{\textit{v-stacks on } \mathrm{Perf}\}; \quad G \mapsto \mathrm{Bun}_G\,.$$

2.3.2. Let $B(G)$ be the set of σ-conjugacy classes in $G(\breve{\mathbb{Q}}_p)$, equipped with the topology coming from the *opposite* of the partial order defined in [30, Section 2.3]. By [37, Theorem 1], there is a homeomorphism

$$|\mathrm{Bun}_G| \to B(G).$$

If μ is a $G(\overline{\mathbb{Q}}_p)$-conjugacy class of minuscule cocharacters, we let $B(G, \mu^{-1}) \subset B(G)$ be the set of μ^{-1}-admissible elements, as defined in [17, Section 1.1.5]; note that this set is closed in the partial order and thus defines an open substack

$$\mathrm{Bun}_{G,\mu^{-1}} \subset \mathrm{Bun}_G,$$

via [31, Proposition 12.9].

2.3.3. If X is an adic space over $\mathrm{Spa}(\mathbb{Z}_p)$, let $X^\diamondsuit$ denote the set-valued functor on Perf given by

$$X^\diamondsuit(S) = \{(S^\sharp, f)\}/\sim$$

for any S in Perf, where $S^\sharp$ is an untilt of S and $f : S^\sharp \to X$ is a morphism of adic spaces. This determines a v-sheaf on Perf by [33, Lem. 18.1.1]. For a Huber pair (A, A^+) over $\mathbb{Z}_p$, we write $\mathrm{Spd}(A, A^+)$ in place of $\mathrm{Spa}(A, A^+)^\diamondsuit$, and when $A^+ = A^\circ$ we write $\mathrm{Spd}(A)$ instead of $\mathrm{Spd}(A, A^+)$. In particular, we see that $\mathrm{Spd}(\mathbb{Z}_p)$ parametrizes isomorphism classes of untilts, cf. [33, Definition 10.1.3].

If $\mathcal{X}$ is a formal scheme over $\operatorname{Spf} \mathbb{Z}_p$, then we can consider it as an adic space over $\operatorname{Spa}(\mathbb{Z}_p)$ using [32, Proposition 2.2.1] and use that adic space to define $\mathcal{X}^{\Diamond}$. This defines a functor from formal schemes over $\mathbb{Z}_p$ to v-sheaves over $\operatorname{Spd} \mathbb{Z}_p$. This functor is fully faithful when restricted to absolutely weakly normal formal schemes that are flat, separated and formally of finite type over $\operatorname{Spf} \mathbb{Z}_p$, see [1, Theorem 2.16].

If X is a scheme over $\operatorname{Spec} \mathbb{Z}_p$, then there are three possible v-sheaves that can be associated to X: There is $X^{\Diamond}$, there is $X^{\Diamond/}$ and there is $X^{\diamond}$, see [1, Definition 2.10] and [27, Definition 2.1.9]. By [27, Corollary 2.1.8], the functor $X \mapsto X^{\Diamond/}$ is fully faithful when restricted to schemes that are flat normal and separated locally of finite type over $\mathbb{Z}_p$.

For schemes X that are locally of finite type over $\operatorname{Spec} \mathbb{Z}_p$, these admit the following descriptions, see [1, Remark 2.11]:

- The v-sheaf $X^{\diamond}$ can be identified with $\left(\widehat{X}\right)^{\Diamond}$, where $\widehat{X}$ is the p-adic formal scheme over $\operatorname{Spf} \mathbb{Z}_p$ given by the completion of X in its special fiber. In [27], they write $X^{\blacklozenge}$ for $X^{\diamond}$.
- The v-sheaf $X^{\Diamond}$ can be identified with $(X^{\mathrm{an}})^{\Diamond}$, where X^{an} is the 'analytification' of X, see [1, Remark 2.11].
- The v-sheaf $X^{\Diamond/}$ is created by gluing $X^{\diamond}$ to $X^{\Diamond}_{\mathbb{Q}_p}$ along the open immersion $X^{\diamond}_{\mathbb{Q}_p} \to X^{\Diamond}_{\mathbb{Q}_p}$.

2.4. **Local Models.** In this section we let G be a connected reductive group over a local field L and μ a $G(\overline{L})$-conjugacy class of minuscule cocharacters of G with reflex field $E \subset \overline{L}$. We fix a parahoric model $\mathcal{G}$ of G over $\mathcal{O}_L$ and we let $\mathcal{G}^{\mathrm{ad}}$ be the corresponding parahoric model for G^{ad}. We let $\operatorname{Gr}_{\mathcal{G}} \to \operatorname{Spd} \mathbb{Z}_p$ be the Beilinson–Drinfeld affine flag variety of $\mathcal{G}$, see [1, Section 4.1], considered as a v-sheaf on Perf. There is a closed subfunctor

$$\operatorname{Gr}_{G,\mu} \subset \operatorname{Gr}_{\mathcal{G}} \times_{\operatorname{Spd} \mathcal{O}_L} \operatorname{Spd} E,$$

see [1, Corollary 4.6], whose closure in $\operatorname{Gr}_{\mathcal{G}} \times_{\operatorname{Spd} \mathcal{O}_L} \operatorname{Spd} \mathcal{O}_E$ defines a closed subfunctor

$$\mathbb{M}^{\mathrm{v}}_{\mathcal{G},\mu} \subset \operatorname{Gr}_{\mathcal{G}} \times_{\operatorname{Spd} \mathcal{O}_L} \operatorname{Spd} \mathcal{O}_E,$$

called the v-sheaf local model attached to $(\mathcal{G},\mu)$. The formation of $\mathbb{M}^{\mathrm{v}}_{\mathcal{G},\mu}$ is functorial for morphisms of pairs $(\mathcal{G}_1,\mu_1) \to (\mathcal{G}_2,\mu_2\})$, and preserves closed embeddings by [1, Proposition 4.16]. Moreover, the morphism

$$\mathbb{M}^{\mathrm{v}}_{\mathcal{G},\mu} \to \mathbb{M}^{\mathrm{v}}_{\mathcal{G}^{\mathrm{ad}},\mu}$$

induced by $(\mathcal{G},\mu) \to (\mathcal{G}^{\mathrm{ad}},\mu)$ is an isomorphism.

2.4.1. Let $L' \subset \overline{L}$ be a finite extension of L and let $E' \supset L'$ be the reflex field of μ considered as a $G(\overline{L})$-conjugacy class of cocharacters of $G_{L'}$.

Lemma 2.4.2. *There is a natural isomorphism*

$$\mathbb{M}^{\mathrm{v}}_{\mathcal{G},\mu} \times_{\operatorname{Spd} \mathcal{O}_E} \operatorname{Spd} \mathcal{O}_{E'} \to \mathbb{M}^{\mathrm{v}}_{\mathcal{G}_{\mathcal{O}_{L'}},\mu}.$$

Proof. The analogous result for

$$\mathrm{Gr}_{\mathcal{G}} \times_{\mathrm{Spd}\,\mathcal{O}_{L'}} \mathrm{Spd}\,\mathcal{O}_{L'} \to \mathrm{Gr}_{\mathcal{G}_{\mathcal{O}_{L'}}}$$

follows directly from the definition, see [1, Section 4.1]. It also follows directly from the definitions that this induces a natural isomorphism

$$\mathrm{Gr}_{G,\mu} \times_{\mathrm{Spa}\,E} \mathrm{Spa}\,E' \to \mathrm{Gr}_{G_{L'},\mu}\,.$$

Since the map $\mathrm{Spec}\,\mathcal{O}_{E'} \to \mathcal{O}_E$ is finite flat, it is in particular proper and open. Thus the map $\mathrm{Spd}\,\mathcal{O}_{E'} \to \mathrm{Spd}\,\mathcal{O}_E$ is partially proper and open. It then follows from [10, Corollary 2.9], that the formation of v-sheaf closures commutes with basechanging from $\mathrm{Spd}\,\mathcal{O}_E$ to $\mathrm{Spd}\,\mathcal{O}_{E'}$. Thus the natural map

$$\mathbb{M}^{\mathrm{v}}_{\mathcal{G},\mu} \times_{\mathrm{Spd}\,\mathcal{O}_E} \mathrm{Spd}\,\mathcal{O}_{E'} \to \mathbb{M}^{\mathrm{v}}_{\mathcal{G}_{\mathcal{O}_{L'}},\mu}.$$

is an isomorphism. □

Now suppose that L'/L is Galois and let Γ be its Galois group.

Lemma 2.4.3. *If L'/L is unramified, then there is a Γ-equivariant isomorphism*

$$\mathbb{M}^{\mathrm{v}}_{\mathcal{H},\mu} \times_{\mathrm{Spd}\,\mathcal{O}_E} \mathrm{Spd}\,\mathcal{O}_{E'} \to \prod_{\gamma\in\Gamma} \mathbb{M}^{\mathrm{v}}_{\mathcal{G},\mu} \times_{\mathrm{Spd}\,\mathcal{O}_E} \mathrm{Spd}\,\mathcal{O}_{E'},$$

under which the natural map from equation 2.4.1 corresponds to the inclusion of the diagonal.

Proof. By Lemma 2.4.2, there is a natural (in particular Γ-equivariant) isomorphism

$$\mathbb{M}^{\mathrm{v}}_{\mathcal{H},\mu} \times_{\mathrm{Spd}\,\mathcal{O}_E} \mathrm{Spd}\,\mathcal{O}_{E'} \to \mathbb{M}^{\mathrm{v}}_{\mathcal{H}_{\mathcal{O}_{L'}},\mu}.$$

There is a Γ-equivariant isomorphism $\mathcal{H}_{\mathcal{O}_{L'}} \to \prod_{\gamma\in\Gamma} \mathcal{G}_{\mathcal{O}'_L}$ since L'/L is unramified. Since the formation of local models commutes with direct products, see [1, Proposition 4.16], this induces a Γ-equivariant isomorphism

$$\mathbb{M}^{\mathrm{v}}_{\mathcal{H}_{\mathcal{O}_{L'}},\mu} \to \prod_{\gamma\in\Gamma} \mathbb{M}^{\mathrm{v}}_{\mathcal{G}_{\mathcal{O}_{L'}},\mu}.$$

Using Lemma 2.4.2 again, we again identify the right-hand side with

$$\prod_{\gamma\in\Gamma} \mathbb{M}^{\mathrm{v}}_{\mathcal{G},\mu} \times_{\mathrm{Spd}\,\mathcal{O}_E} \mathrm{Spd}\,\mathcal{O}_{E'}.$$

Moreover, under these identifications the natural map

$$\mathbb{M}^{\mathrm{v}}_{\mathcal{G},\mu} \to \mathbb{M}^{\mathrm{v}}_{\mathcal{H},\mu}$$

corresponds to the inclusion of the diagonal into the product. □

2.4.4. Now let $\operatorname{Spec}\mathcal{O}_{L'} \to \operatorname{Spec}\mathcal{O}_L$ be a finite étale Galois cover with Galois group Γ and generic fiber $\operatorname{Spec} L' \to \operatorname{Spec} L$. Define $\mathcal{H} := \operatorname{Res}_{\mathcal{O}_{L'}/\mathcal{O}_L}\mathcal{G}$ and let μ be the induced conjugacy class of cocharacters of $H := \mathcal{H}_L$. Then Γ-acts on $\mathcal{H}$ in a way that preserves μ and thus acts on $\mathbb{M}^{\mathrm{v}}_{\mathcal{H},\mu}$. Moreover the natural map $(\mathcal{G},\mu) \to (\mathcal{H},\mu)$ induces a natural map

$$\mathbb{M}^{\mathrm{v}}_{\mathcal{G},\mu} \to \mathbb{M}^{\mathrm{v}}_{\mathcal{H},\mu}, \tag{2.4.1}$$

which is Γ-equivariant for the trivial Γ-action on the source. Let $\mathfrak{p}$ be a maximal ideal of $\mathcal{O}_{L'}$, let $\mathcal{O}_{L''}$ be the local ring of $\mathcal{O}_{L'}$ at $\mathfrak{p}$ with fraction field L''. Let $\Gamma'' \subset \Gamma$ be the stabilizer of $\mathfrak{p}$, which is also the Galois group of L''/L. Choose an embedding $L'' \to \overline{L}$ and let E'' be the reflex field of μ considered as an $G_{L''}(\overline{L})$-conjugacy class of cocharacters of $G_{L''}$.

Lemma 2.4.5. *There is a Γ-equivariant isomorphism*

$$\mathbb{M}^{\mathrm{v}}_{\mathcal{H},\mu} \times_{\operatorname{Spd}\mathcal{O}_E} \operatorname{Spd}\mathcal{O}_{E''} \to \prod_{\gamma\in\Gamma} \mathbb{M}^{\mathrm{v}}_{\mathcal{G},\mu} \times_{\operatorname{Spd}\mathcal{O}_E} \operatorname{Spd}\mathcal{O}_{E'},$$

under which the natural map from equation 2.4.1 corresponds to the inclusion of the diagonal.

Proof. Since $\operatorname{Spec}\mathcal{O}_{L'} \to \operatorname{Spec}\mathcal{O}_L$ is Galois there is a Γ-equivariant isomorphism

$$\operatorname{Spec}\mathcal{O}_{L'} \to \hom_{\Gamma''}(\Gamma, \operatorname{Spec}\mathcal{O}_{L''}).$$

Similarly there is a Γ-equivariant isomorphism

$$\mathcal{H} \to \hom_{\Gamma''}(\Gamma, \operatorname{Res}_{\mathcal{O}_{L''}/\mathbb{Z}_p}\mathcal{G}_{\mathcal{O}_{L''}}) \simeq \prod_{\Gamma''\backslash\Gamma} \operatorname{Res}_{\mathcal{O}_{L''}/\mathbb{Z}_p}\mathcal{G}_{\mathcal{O}_{L''}}).$$

Since the formation of local models commutes with products by [10, Corollary 2.9], the lemma reduces to Lemma 2.4.3. □

2.4.6. Recall that there are unique (up to unique isomorphism) flat and (absolutely) weakly normal schemes $\mathbb{M}_{\mathcal{G},\mu}$ over $\mathcal{O}_E$ with associated v-sheaf isomorphic to $\mathbb{M}^{\mathrm{v}}_{\mathcal{G},\mu}$, by [1, Theorem 1.11].

Proposition 2.4.7. *The natural map of equation* (2.4.1) *induces an isomorphism*

$$\mathbb{M}_{\mathcal{G},\mu} \to \mathbb{M}^{\Gamma}_{\mathcal{H},\mu}.$$

Proof. By [1, Proposition 2.18], the functor sending a proper flat absolutely weakly normal scheme over $\operatorname{Spd}\mathcal{O}_L$ to its associated v-sheaf is fully faithful. This functor moreover commutes with base change along $\operatorname{Spec}\mathcal{O}_{E''} \to \operatorname{Spec}\mathcal{O}_E$ and with fiber products. Thus by Lemma 2.4.5 there is a Γ-equivariant isomorphism

$$\mathbb{M}_{\mathcal{H},\mu} \times_{\operatorname{Spec}\mathcal{O}_E} \operatorname{Spec}\mathcal{O}_{E''} \to \prod_{\gamma\in\Gamma} \mathbb{M}_{\mathcal{G},\mu} \times_{\operatorname{Spec}\mathcal{O}_E} \operatorname{Spec}\mathcal{O}_{E''}$$

under which the natural map $\mathbb{M}_{\mathcal{G},\mu} \to \mathbb{M}_{\mathcal{H},\mu}$ corresponds to the inclusion of the diagonal. The proposition follows immediately from this description. □

Corollary 2.4.8. *The natural map*

$$[\mathbb{M}_{\mathcal{G},\mu}/\mathcal{G}] \to [\mathbb{M}_{\mathcal{H},\mu}/\mathcal{H}]^{h\Gamma}$$

is an isomorphism.

Proof. After basechanging to $\operatorname{Spa}(\mathcal{O}_L)$ we have an isomorphism $\mathcal{H} \simeq \hom(\Gamma, \mathcal{G})$ and thus it follows from Lemma 2.2.1 that $\underline{H}^1(\Gamma, \mathcal{H})$ vanishes. The corollary now follows from Proposition 2.4.7 and Corollary A.2.9. $\square$

2.5. **Shtukas.** In this section we let G be a connected reductive group over $\mathbb{Q}_p$ and μ a $G(\overline{\mathbb{Q}}_p)$-conjugacy class of minuscule cocharacters of G with reflex field $L \subset E \subset \overline{\mathbb{Q}}_p$. We fix a parahoric model $\mathcal{G}$ of G over $\mathbb{Z}_p$.

Recall from [27, Definition 2.4.3] that a $\mathcal{G}$-shtuka over a perfectoid space S with leg at an untilt $S^\sharp =$ is defined to be a quadruple: $(\mathcal{Q}, \mathcal{P}, \phi_{\mathcal{P}}, \kappa)$ where $\mathcal{Q}$ and $\mathcal{P}$ are $\mathcal{G}$-torsors over the analytic adic space $S \dot{\times} \mathbb{Z}_p$, where $\kappa : \mathcal{Q} \to (\operatorname{Frob}_S \times 1)^* \mathcal{P}$ is an isomorphism of $\mathcal{G}$-torsors and where

$$\phi_{\mathcal{P}} : \mathcal{Q}\big|_{S \dot{\times} \mathbb{Z}_p \setminus S^\sharp} \to \mathcal{P}\big|_{S \dot{\times} \mathbb{Z}_p \setminus S^\sharp}$$

is an isomorphism of $\mathcal{G}$-torsors over $S \dot{\times} \mathbb{Z}_p \setminus S^\sharp$. Here $S^\sharp \subset S \dot{\times} \mathbb{Z}_p$ is the Cartier divisor coming from the untilt $S^\sharp$. To be precise, here we consider $(\operatorname{Frob}_S \times 1)^* \mathcal{P}$ as a$\mathcal{G}$-torsor via the isomorphism $(\operatorname{Frob}_S \times 1)^* \mathcal{G} \to \mathcal{G}$ coming from the fact that $\mathcal{G}$ is defined over $\mathbb{Z}_p$. Note that the data of $\mathcal{Q}$ and κ is superfluous in our definition, but it will be useful for us later to keep track of this information. In Section 4 we will often omit $\mathcal{Q}$ and κ from the notation

2.5.1. Let $\operatorname{Perf}_{\mathbb{Z}_p}$ be the category of perfectoid spaces S of characteristic p equipped with a map $S^\lozenge \to \operatorname{Spd} \mathbb{Z}_p$, equipped with its v-topology. Let $\operatorname{Sht}_{\mathcal{G}}$ be the stack of $\mathcal{G}$-shtukas over $\operatorname{Perf}_{\mathbb{Z}_p}$, considered as a category fibered in groupoids over $\operatorname{Perf}_{\mathbb{Z}_p}$. Explicitly, this means that it is the category whose objects are quintuples $(S \to \operatorname{Spd} \mathbb{Z}_p, \mathcal{Q}, \mathcal{P}, \phi_{\mathcal{P}}, \kappa)$, where $S \to \operatorname{Spd} \mathbb{Z}_p$ is an object of $\operatorname{Perf}_{\mathbb{Z}_p}$ and $(\mathcal{Q}, \mathcal{P}, \phi_{\mathcal{P}}, \kappa)$ is a $\mathcal{G}$-shtuka over S. A morphism $f : (S \to \operatorname{Spd} \mathbb{Z}_p, \mathcal{Q}, \mathcal{P}, \phi_{\mathcal{P}}, \kappa) \to (S' \to \operatorname{Spd} \mathbb{Z}_p, \mathcal{Q}', \mathcal{P}', \phi_{\mathcal{P}'}, \kappa')$ is a triple $(f, f_{\mathcal{P}}, f_{\mathcal{Q}})$ consisting of a morphism $f : S \to S'$ and $\mathcal{G}$-equivariant morphisms $f_{\mathcal{P}} : \mathcal{P} \to \mathcal{P}', f_{\mathcal{Q}} : \mathcal{Q} \to \mathcal{Q}'$ fitting in a pair of Cartesian diagrams

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{f_{\mathcal{P}}} & \mathcal{P}' \\ \downarrow & & \downarrow \\ S \dot{\times} \mathbb{Z}_p & \xrightarrow{f \times 1} & S' \dot{\times} \mathbb{Z}_p \end{array} \qquad \begin{array}{ccc} \mathcal{Q} & \xrightarrow{f_{\mathcal{Q}}} & \mathcal{Q}' \\ \downarrow & & \downarrow \\ S \dot{\times} \mathbb{Z}_p & \xrightarrow{f \times 1} & S' \dot{\times} \mathbb{Z}_p \end{array}$$

such that the following diagrams commute

$$\begin{array}{ccc} \mathcal{Q}\big|_{S \dot{\times} \mathbb{Z}_p \setminus S^\sharp} & \xrightarrow{\phi_{\mathcal{P}}} & \mathcal{P}\big|_{S \dot{\times} \mathbb{Z}_p \setminus S^\sharp} \\ \downarrow{\scriptstyle f_{\mathcal{P}}} & & \downarrow{\scriptstyle f_{\mathcal{Q}}} \\ \mathcal{P}\big|_{S' \times \mathbb{Z}_p \setminus S^\sharp} & \xrightarrow{\phi_{\mathcal{P}'}} & \mathcal{Q}'\big|_{S' \times \mathbb{Z}_p \setminus S^\sharp} \end{array} \qquad \begin{array}{ccc} \mathcal{Q} & \xrightarrow{\kappa} & (\operatorname{Frob}_S \times 1)^* \mathcal{P} \\ \downarrow{\scriptstyle f_{\mathcal{Q}}} & & \downarrow \\ \mathcal{Q}' & \xrightarrow{\kappa'} & (\operatorname{Frob}_{S'} \times 1)^* \mathcal{P}', \end{array}$$

where the right vertical arrow in the second diagram is the arrow induced by $f_{\mathcal{P}}$. Since μ is defined over E, there is a closed substack[1]

$$\mathrm{Sht}_{\mathcal{G}} \subset \mathrm{Sht}_{\mathcal{G}} \otimes_{\mathrm{Spd}\,\mathbb{Z}_p} \mathrm{Spd}\,\mathcal{O}_E,$$

defined as those shtukas where $\phi_{\mathcal{P}}$ has 'relative position bounded by' the v-sheaf local model

$$\mathbb{M}^{\mathrm{v}}_{\mathcal{G},\mu} \subset \mathrm{Gr}_{\mathcal{G}} \otimes_{\mathrm{Spd}\,\mathbb{Z}_p} \mathrm{Spd}\,\mathcal{O}_E,$$

see [27, Definition 2.4.4]. It is clear that a homomorphism $f : \mathcal{G} \to \mathcal{G}'$ of parahoric group schemes induces a morphism.

$$f : \mathrm{Sht}_{\mathcal{G}} \to \mathrm{Sht}_{\mathcal{G}'}$$

by pushing out torsors. Noting that the reflex field E' of μ' is contained in E, this restricts to a morphism

$$\mathrm{Sht}_{\mathcal{G}} \times_{\mathrm{Spd}\,\mathcal{O}_E} \mathrm{Spd}\,\mathcal{O}_{E'} \to \mathrm{Sht}_{\mathcal{G}',\mu'},$$

if $f(\mu) = \mu'$. This upgrades to a 2-categorical statement as follows: Fix a field $L \subset \overline{\mathbb{Q}}_p$ and let ShtPr_L be the category of pairs $(\mathcal{G}, \mu)$ such that the reflex field $E(\mu)$ of μ is contained in L, and where the morphisms are the obvious morphisms of pairs. We will often write E for $E(\mu)$ if μ is clear from the context. We let $\mathcal{D}^{\circ}_L$ be the strict $(2,1)$-category of categories fibered in groupoids over $\mathrm{Perf}_{\mathcal{O}_L}$ and $\mathcal{D}_L$ the full subcategory of categories fibered in groupoids over Perf_L.

Lemma 2.5.2. *There is a weak functor* $\mathrm{Sht} : \mathrm{ShtPr}_L \to \mathcal{D}^{\circ}_L$*, which on objects sends* $(\mathcal{G}, \mu)$ *to* $\mathrm{Sht}_{\mathcal{G},\mu} \times_{\mathrm{Spd}\,\mathcal{O}_E} \mathrm{Spd}\,\mathcal{O}_L$ *and which sends a morphism* $(\mathcal{G}, \mu) \to (\mathcal{G}', \mu')$ *to the induced morphism* $\mathrm{Sht}_{\mathcal{G},\mu} \times_{\mathrm{Spd}\,\mathcal{O}_E} \mathrm{Spd}\,\mathcal{O}_L \to \mathrm{Sht}_{\mathcal{G},\mu} \times_{\mathrm{Spd}\,\mathcal{O}_{E'}} \mathrm{Spd}\,\mathcal{O}_L$ *induced by pushing out torsors along* $\mathcal{G} \to \mathcal{G}'$.

Proof. A morphism $f : \mathcal{G} \to \mathcal{G}'$ induces a morphism $\mathrm{Sht}_{\mathcal{G}} \to \mathrm{Sht}_{\mathcal{G}'}$ by sending $(S \to \mathrm{Spd}\,\mathbb{Z}_p, \mathcal{Q}, \mathcal{P}, \phi_{\mathcal{P}}, \kappa)$ to $(S \to \mathrm{Spd}\,\mathbb{Z}_p, \mathcal{Q} \times^{\mathcal{G}} \mathcal{G}', \mathcal{P} \times^{\mathcal{G}} \mathcal{G}', \phi_{\mathcal{P} \times^{\mathcal{G}} \mathcal{G}'}, \kappa')$. Here κ' is the unique isomorphism induced by κ, using the fact that $f : \mathcal{G} \to \mathcal{G}'$ is defined over $\mathbb{Z}_p$ and thus commutes with Frob_S. To turn this into a weak functor, we use the coherence data for pushing out torsors coming from the proof of Lemma A.2.1.

As discussed above, this morphism restricts to a morphism between $\mathrm{Sht}_{\mathcal{G},\mu}$ and $\mathrm{Sht}_{\mathcal{G}',\mu'}$ and thus induces a morphism $\mathrm{Sht}_{\mathcal{G},\mu} \times_{\mathrm{Spd}\,\mathcal{O}_E} \mathrm{Spd}\,\mathcal{O}_L \to \mathrm{Sht}_{\mathcal{G}',\mu'} \times_{\mathrm{Spd}\,\mathcal{O}_{E'}} \mathrm{Spd}\,\mathcal{O}_L$. □

2.6. **Integral local Shimura varieties.** In this section we let G be a connected reductive group over $\mathbb{Q}_p$ and μ a $G(\overline{\mathbb{Q}}_p)$-conjugacy class of minuscule cocharacters of G with reflex field $E \subset \overline{\mathbb{Q}}_p$. We fix a parahoric model $\mathcal{G}$ of G over $\mathbb{Z}_p$. It follows from Lemma A.2.1 that there is weak functor Bun from the category of reductive groups over $\mathbb{Q}_p$ to the strict $(2,1)$-category of v-stacks on Perf. Recall that there is a faithful morphism $\mathrm{BL}^{\circ} : \mathrm{Sht}_{\mathcal{G},\mu} \to \mathrm{Bun}_G$, as explained in [27, Section 2.2.2], see [4, Section 2.1.12].

Lemma 2.6.1. *There is a weak natural transformation* $\mathrm{Sht} \to \mathrm{Bun}$ *of weak functors* $\mathrm{ShtPr}_L \to \mathrm{Stk}_{\mathrm{Perf}}$*, giving* $\mathrm{BL}^{\circ} : \mathrm{Sht}_{\mathcal{G},\mu} \to \mathrm{Bun}_G$ *on objects.*

[1]Here by $\mathrm{Sht}_{\mathcal{G}} \otimes_{\mathrm{Spd}\,\mathbb{Z}_p} \mathrm{Spd}\,\mathcal{O}_E$ we mean the category fibered in groupoids over $\mathrm{Perf}_{\mathcal{O}_E}$ given by the fiber product construction of [36, Lemma 0040].

Proof. This comes down to checking that pushing out torsors is compatible with descending them from $\mathcal{Y}_{[r,\infty)}(S)$ (notation as in [27, Section 2.2.2]) to X_S, which is straightforward. □

For $b : \operatorname{Spd} \overline{\mathbb{F}}_p \to \operatorname{Sht}_{\mathcal{G},\mu}$ we define $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}$ as the 2-fiber product

$$\begin{array}{ccc} \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu} & \longrightarrow & \operatorname{Sht}_{\mathcal{G},\mu} \\ \downarrow & & \downarrow \\ \operatorname{Spd} \overline{\mathbb{F}}_p & \xrightarrow{b} & \operatorname{Bun}_G . \end{array}$$

Note that a morphism of triples $(\mathcal{G}, \mu, b) \to (\mathcal{G}', \mu', b')$ induces a morphism $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu} \to \mathcal{M}^{\mathrm{int}}_{\mathcal{G}',b',\mu'}$ taking x_0 to x_0'. It is straightforward to check that this gives a functor from the category of triple $(\mathcal{G}, \mu, b)$ with reflex field contained in L, to $\mathcal{D}^{\circ}$, sending

$$(\mathcal{G}, \mu, b) \mapsto \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu} \otimes_{\operatorname{Spd} \mathcal{O}_E} \operatorname{Spd} \mathcal{O}_L.$$

2.6.2. *Fixed points of integral local Shimura varieties.* Let $\operatorname{Spec} \mathcal{O}_L \to \operatorname{Spec} \mathbb{Z}_p$ be a finite étale Galois cover with Galois group Γ and generic fiber $\operatorname{Spec} L \to \operatorname{Spec} \mathbb{Q}_p$. Define $\mathcal{H} := \operatorname{Res}_{\mathcal{O}_L/\mathbb{Z}_p} \mathcal{G}_{\mathcal{O}_L}$ and let μ be the induced conjugacy class of cocharacters of $H := \mathcal{H}_{\mathbb{Q}_p}$. Then Γ-acts on $\mathcal{H}$ in a way that preserves μ and thus acts on $\operatorname{Sht}_{\mathcal{H},\mu} \to \operatorname{Perf}_{\mathcal{O}_E}$ by Lemma 2.5.2.

Lemma 2.6.3. *Let X be an adic space over $\operatorname{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$, then cocycles $\sigma : \Gamma \to \mathcal{H}(X)$ are trivial étale locally on X.*

Proof. After basechanging to $\operatorname{Spa}(\mathcal{O}_L, \mathcal{O}_L)$, we have an isomorphism $\mathcal{H} \simeq \hom(\Gamma, \mathcal{G})$ and the result then follows from Lemma 2.2.1. □

Corollary 2.6.4. *Consider the stack $\mathbb{B}\mathcal{H}$ on the category of adic spaces over $\operatorname{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$ equipped with the étale topology. Then the natural map*

$$\mathbb{B}\mathcal{G} \to (\mathbb{B}\mathcal{H})^{h\Gamma}$$

is an equivalence.

Proof. This is a direct consequence of Lemma 2.6.3 and Proposition A.2.7. □

2.6.5. *Homotopy fixed points of shtukas.* There is a natural map $\operatorname{Sht}_{\mathcal{G}} \to \operatorname{Sht}_{\mathcal{H}}^{h\Gamma}$ of categories fibered in groupoids over $\operatorname{Perf}_{\mathbb{Z}_p}$; this follows from Lemma 2.5.2.

Lemma 2.6.6. *The natural map*

$$\operatorname{Sht}_{\mathcal{G}} \to \operatorname{Sht}_{\mathcal{H}}^{h\Gamma}$$

is an isomorphism.

Proof. To prove that this is an equivalence, it suffices to this fiberwise over $\operatorname{Spd} \mathbb{Z}_p$; so fix $S \to \operatorname{Spd} \mathbb{Z}_p$ corresponding to an untilt $S^\sharp$ of S. The category of $\mathcal{H}$-shtukas over S can be described as the 2-fiber product

$$\begin{array}{ccc} \mathrm{Sht}_{\mathcal{H}}(S) & \longrightarrow & \mathbb{B}\mathcal{H}(S \dot{\times} \mathbb{Z}_p) \\ \downarrow & & \downarrow \Gamma_{\mathrm{Frob}_S} \\ \mathbb{B}\mathcal{H}(S \dot{\times} \mathbb{Z}_p) & \xrightarrow{\Delta} & \mathbb{B}\mathcal{H}(S \dot{\times} \mathbb{Z}_p \setminus S^\sharp) \times \mathbb{B}\mathcal{H}(S \dot{\times} \mathbb{Z}_p), \end{array}$$

where Δ is the diagonal and where Γ_{Frob_S} is the graph of pullback along Frob_S. This diagram is Γ-equivariant because the right vertical and bottom horizontal maps are equivariant by Lemma A.2.1. Lemma A.1.11 tells us that the homotopy fixed points of the diagram give a 2-Cartesian diagram

$$\begin{array}{ccc} \mathrm{Sht}_{\mathcal{H}}(S)^{h\Gamma} & \longrightarrow & \mathbb{B}\mathcal{H}(S \dot{\times} \mathbb{Z}_p)^{h\Gamma} \\ \downarrow & & \downarrow \Gamma_{\mathrm{Frob}_S} \\ \mathbb{B}\mathcal{H}(S \dot{\times} \mathbb{Z}_p)^{h\Gamma} & \xrightarrow{\Delta} & \mathbb{B}\mathcal{H}(S \dot{\times} \mathbb{Z}_p \setminus S^\sharp)^{h\Gamma} \times \mathbb{B}\mathcal{H}(S \dot{\times} \mathbb{Z}_p)^{h\Gamma}, \end{array}$$

which after repeatedly applying Corollary 2.6.4 can be identified with the 2-Cartesian diagram

$$\begin{array}{ccc} \mathrm{Sht}_{\mathcal{H}}(S)^{h\Gamma} & \longrightarrow & \mathbb{B}\mathcal{G}(S \dot{\times} \mathbb{Z}_p) \\ \downarrow & & \downarrow \Gamma_{\mathrm{Frob}_S} \\ \mathbb{B}\mathcal{G}(S \dot{\times} \mathbb{Z}_p) & \xrightarrow{\Delta} & \mathbb{B}\mathcal{G}(S \dot{\times} \mathbb{Z}_p \setminus S^\sharp) \times \mathbb{B}\mathcal{G}(S \dot{\times} \mathbb{Z}_p). \end{array}$$

By construction, the map $\mathrm{Sht}_{\mathcal{G}}$ to $\mathrm{Sht}_{\mathcal{H}}(S)^{h\Gamma}$ compatible is compatible with the 2-fiber product description, and thus an isomorphism. □

2.6.7. By abuse of notation we also use μ to denote the induced $H(\overline{\mathbb{Q}}_p)$-conjugacy class of cocharacters of H, and we will later do this for all other groups.

Lemma 2.6.8. *The natural map*

$$\mathrm{Sht}_{\mathcal{G},\mu} \to \mathrm{Sht}_{\mathcal{H},\mu}^{h\Gamma}$$

is an isomorphism.

Proof. This follows from Lemma 2.6.6 as soon as we can prove that a $\mathcal{G}$-shtuka whose induced $\mathcal{H}$-shtuka is bounded by μ is itself bounded by μ. This follows from the fact that the natural isomorphism

$$\mathrm{Gr}_{\mathcal{G}} \to \mathrm{Gr}_{\mathcal{H}}^{\Gamma}$$

induces an isomorphism $\mathbb{M}^{\mathrm{v}}_{\mathcal{G},\mu} \to \mathbb{M}^{\mathrm{v},\Gamma}_{\mathcal{H},\mu}$, see Proposition 2.4.7. □

Lemma 2.6.9. *Let* $\operatorname{Spec} L \to \operatorname{Spec} \mathbb{Q}_p$ *be a finite étale Galois cover (not necessarily unramified) with Galois group* Γ*. Then the natural map*

$$\mathrm{Bun}_G \to \mathrm{Bun}_H^{h\Gamma}$$

is an isomorphism.

Proof. It follows from the proof of Corollary 2.6.4 that

$$\mathbb{B}G \to (\mathbb{B}H)^{h\Gamma}$$

is an equivalence on the category of adic spaces over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. Indeed, the point is that we can use Lemma 2.2.1 after the finite étale cover $\mathrm{Spa}(L, \mathcal{O}_L) \to \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ and then conclude using Corollary A.2.9. □

Now we return to the assumption that $\mathrm{Spec}\, \mathcal{O}_L \to \mathrm{Spec}\, \mathbb{Z}_p$ is a finite étale cover with Galois group Γ. If $b : \mathrm{Spd}\, \overline{\mathbb{F}}_p \to \mathrm{Sht}_{\mathcal{H},\mu}$ is induced by $b : \mathrm{Spd}\, \overline{\mathbb{F}}_p \to \mathrm{Sht}_{\mathcal{G},\mu}$, then there is an action of Γ on $\mathcal{M}^{\mathrm{int}}_{\mathcal{H},b,\mu}$.

Proposition 2.6.10. *The natural map* $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu} \to \left(\mathcal{M}^{\mathrm{int}}_{\mathcal{H},b,\mu}\right)^{\Gamma}$ *is an isomorphism.*

Proof. The Cartesian diagram defining $\mathcal{M}^{\mathrm{int}}_{\mathcal{H},b,\mu}$ is Γ-equivariant by Lemma 2.6.1 and the fact that b is induced by $b : \mathrm{Spd}\, \overline{\mathbb{F}}_p \to \mathrm{Sht}_{\mathcal{G},\mu}$. The Proposition then follows by taking Γ-homotopy fixed points of this diagram, and using Lemma A.1.11 in combination with Lemmas 2.6.8 and 2.6.9. □

3. Fixed points of Shimura varieties

Let (G, X) be a Shimura datum and let F be a totally real Galois extension of $\mathbb{Q}$ with Galois group Γ. Let $H := \mathrm{Res}_{\mathsf{F}/\mathbb{Q}}\, G_F$ equipped with its natural action of Γ. Following a suggestion of Rapoport, we call (H, Y) the *Piatetski-Shapiro construction* for (G, X) associated to F.[2]For $h : \mathbb{S} \to \mathsf{G}_{\mathbb{R}}$ in X we let Y be the $\mathsf{H}(\mathbb{R})$-conjugacy class of the composition $h : \mathbb{S} \to G_{\mathbb{R}} \to \mathsf{H}_{\mathbb{R}}$, this does not depend on the choice of h. Then (H, Y) is a Shimura datum and the action of Γ on H is by automorphisms of Shimura data. Moreover, the natural closed immersion $\mathsf{G} \to \mathsf{H}$ is a morphism of Shimura data. The goal of this section is to investigate the fixed points of the action of Γ on the Shimura varieties for (H, Y), and to compare them to the Shimura varieties for (G, X)

3.1. The case of tori.

As a toy example, we study the case when $\mathsf{G} = \mathsf{T}$ is a torus with $\mathbb{R}$-split rank zero. Then the $\mathbb{R}$-split rank of $\mathsf{H} = \mathrm{Res}_{\mathsf{F}/\mathbb{Q}}\, \mathsf{T}_F$ is also zero, and thus for neat $K \subset \mathsf{H}(\mathbb{A}_f)$ the natural map

$$\mathsf{H}(\mathbb{Q}) \to \mathsf{H}(\mathbb{A}_f)/K$$

is injective by Lemma 3.2.1. The cokernel of this injection defines the set of $\mathbb{C}$-points of the Shimura variety $\mathbf{Sh}_K(\mathsf{H}, \mathsf{Y})$. If K is Γ-stable, then Γ acts on $\mathbf{Sh}_K(\mathsf{H}, \mathsf{Y})$ and there is a long exact sequence

$$0 \to K^{\Gamma} \to \mathsf{G}(\mathbb{A}_f^p) \to (\mathsf{H}(\mathbb{A}_f)/K))^{\Gamma} \to H^1(\Gamma, K) \to H^1(\Gamma, \mathsf{H}(\mathbb{A}_f)) \to \cdots$$

[2]This construction was used by Borovoi [2] and Milne [25] in their proof of the conjecture of Langlands on conjugation of Shimura varieties, and they credit it to Piatetski-Shapiro.

We will see in Proposition 3.4.7 that it is possible to find arbitrary small Γ-stable K with $H^1(\Gamma, K) = 0$, at least if F is tamely ramified over $\mathbb{Q}$. So from now on we will assume that $H^1(\Gamma, K) = 0$, which implies that the natural map

$$\mathsf{G}(\mathbb{A}_f)/K^{\Gamma} \to (\mathsf{H}(\mathbb{A}_f)/K))^{\Gamma}.$$

is an isomorphism and that the natural map

$$H^1(\Gamma, \mathsf{H}(\mathbb{A}_f)/K) \to H^1(\Gamma, \mathsf{H}(\mathbb{A}_f))$$

is injective. It moreover implies that there is a long exact sequence

$$0 \to \mathsf{G}(\mathbb{Q}) \to \mathsf{G}(\mathbb{A}_f)/K^{\Gamma} \to \mathbf{Sh}_K(\mathsf{H}, \mathsf{Y})^{\Gamma}(\mathbb{C}) \to H^1(\Gamma, \mathsf{H}(\mathbb{Q})) \to H^1(\Gamma, \mathsf{H}(\mathbb{A}_f)/K).$$

Thus the obstruction to the natural map

$$\mathbf{Sh}_{K^{\Gamma}}(\mathsf{G}, \mathsf{X})(\mathbb{C}) \to \mathbf{Sh}_K(\mathsf{H}, \mathsf{Y})^{\Gamma}(\mathbb{C})$$

being a bijection is given by the kernel of

$$H^1(\Gamma, \mathsf{H}(\mathbb{Q})) \to H^1(\Gamma, \mathsf{H}(\mathbb{A}_f)).$$

We will later prove, see Lemma 3.3.1, that this can be identified with the kernel of

$$Ш^1(\mathbb{Q}, \mathsf{G}) \to Ш^1(\mathsf{F}, G),$$

in particular, the obstruction is finite. In the rest of the section, we will prove that the same result, suitably interpreted, holds for arbitrary Shimura varieties.

3.2. **Shimura varieties and Shimura stacks.** Let the notation be as in the start of Section 3 and let $K \subset \mathsf{H}(\mathbb{A}_f^p)$ be a compact open subgroup. It turns out that it is easier to analyze the Γ-homotopy fixed points of the stack quotient/action groupoid

$$\left[\mathsf{H}(\mathbb{Q})\backslash\left(\mathsf{Y} \times \mathsf{H}(\mathbb{A}_f^p)/K\right)\right], \tag{3.2.1}$$

than it is to analyze the Γ-fixed points of the Shimura variety $\mathbf{Sh}_K(\mathsf{H}, \mathsf{Y})$ with $\mathbb{C}$-points

$$\mathbf{Sh}_K(\mathsf{H}, \mathsf{Y})(\mathbb{C}) = \mathsf{H}(\mathbb{Q})\backslash\left(\mathsf{Y} \times \mathsf{H}(\mathbb{A}_f^p)/K\right).$$

Thus it is important for us to understand when the Shimura variety for (H, Y) is equal to the stack quotient in (3.2.1) (for sufficiently small K).

Recall Milne's axiom SV5 for a Shimura datum (G, X) from [26, p. 64]; it asks that $Z(\mathbb{Q}) \subset Z(\mathbb{A}_f)$ is discrete, where $Z = Z_{\mathsf{G}}$ is the center of G. By [18, Lemma 1.5.5] this happens if and only if the $\mathbb{Q}$-split rank of Z is equal to the $\mathbb{R}$-split rank of Z. The following lemma is well known [26, Prop. 3.1, Lem. 5.13]

Lemma 3.2.1. *If axiom SV5 holds for* (G, X)*, then for neat K the group* $\mathsf{G}(\mathbb{Q})$ *acts freely on* $\mathsf{X} \times \mathsf{G}(\mathbb{A}_f^p)/K$.

Unfortunately if SV5 holds for (G, X) then it often does not hold for (H, Y). For example if $Z_{\mathsf{G}} = \mathbb{G}_m$, then $Z_{\mathsf{H}} = \mathrm{Res}_{\mathsf{F}/\mathbb{Q}} \mathbb{G}_m$, which has $\mathbb{Q}$-split rank one but $\mathbb{R}$-split rank equal to $[F : \mathbb{Q}]$. We make this observation precise in the following lemma.

Lemma 3.2.2. *Suppose that* (G,X) *satisfies SV5 and that* $[F:\mathbb{Q}]>1$. *Then* (H,Y) *satisfies SV5 if and only if the* $\mathbb{R}$*-split rank of* Z_{G} *is zero.*

Proof. By [18, Lemma 1.5.5], there is an isogeny $Z_{\mathsf{G}} \sim \mathsf{T}_1 \times T_2$ where T_1 is $\mathbb{Q}$-split and T_2 has $\mathbb{R}$-rank zero. This induces an isogeny

$$Z_{\mathsf{H}} \simeq \mathrm{Res}_{\mathsf{F}/\mathbb{Q}} \mathsf{T}_{1,\mathsf{F}} \times \mathrm{Res}_{\mathsf{F}/\mathbb{Q}} \mathsf{T}_{2,\mathsf{F}}.$$

The torus $\mathrm{Res}_{\mathsf{F}/\mathbb{Q}} \mathsf{T}_{2,\mathsf{F}}$ still has $\mathbb{R}$-rank zero. The torus $\mathrm{Res}_{\mathsf{F}/\mathbb{Q}} \mathsf{T}_{1,\mathsf{F}}$ has $\mathbb{Q}$-rank equal to the $\mathbb{Q}$-split rank d of Z_{G}, and has $\mathbb{R}$-split rank equal to $d \cdot [F:\mathbb{Q}]$. Thus the $\mathbb{Q}$-split rank of Z_{H} is equal to the $\mathbb{R}$-split rank of Z_{H} if and only if $d=0$. □

3.3. **Fixed points of adelic quotients.** Let the notation be as in the start of Section 3. Before we investigate the homotopy fixed points of the groupoid (3.2.1), we first investigate the fixed points of $\mathsf{H}(\mathbb{Q})\backslash \mathsf{Y} \times \mathsf{H}(\mathbb{A}_f)$. To understand the Γ-fixed points of the quotient $\mathsf{H}(\mathbb{Q})\backslash \mathsf{H}(\mathbb{A})$, we need to understand the map $H^1(\Gamma, \mathsf{H}(\mathbb{Q})) \to H^1(\Gamma, \mathsf{H}(\mathbb{A}))$. For this we will use the inflation maps

$$H^1(\Gamma, \mathsf{H}(\mathbb{Q})) = H^1(\Gamma, \mathsf{G}(\mathsf{F})) \to H^1(\mathrm{Gal}_{\mathbb{Q}}, \mathsf{G}(\overline{\mathbb{Q}})) =: H^1(\mathbb{Q}, \mathsf{G})$$
$$H^1(\Gamma, \mathsf{H}(\mathbb{A})) = H^1(\Gamma, \mathsf{G}(\mathbb{A}_{\mathsf{F}})) \to H^1(\mathrm{Gal}_{\mathbb{Q}}, \mathsf{G}(\overline{\mathbb{A}})),$$

where $\overline{\mathbb{A}} = \mathbb{A} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ and $\mathbb{A}_{\mathsf{F}}) = \mathbb{A} \otimes_{\mathbb{Q}} \mathsf{F}$. As explained on [29, p. 298], there is an injective map

$$H^1(\mathrm{Gal}_{\mathbb{Q}}, \mathsf{G}(\overline{\mathbb{A}})) \to \prod_v H^1(\mathrm{Gal}_{\mathbb{Q}_v}, \mathsf{G}(\overline{\mathbb{Q}}_v)),$$

such that the induced map $H^1(\mathrm{Gal}_{\mathbb{Q}}, \mathsf{G}(\overline{\mathbb{Q}})) \to \prod_v H^1(\mathrm{Gal}_{\mathbb{Q}_v}, \mathsf{G}(\overline{\mathbb{Q}}_v))$ is the natural map.

Lemma 3.3.1. *The inflation map induces a bijection*

$$\ker\left(H^1(\Gamma, \mathsf{H}(\mathbb{Q})) \to H^1(\Gamma, \mathsf{H}(\mathbb{A}))\right) \to \ker\left(Ш^1(\mathbb{Q}, \mathsf{G}) \to Ш^1(\mathsf{F}, G)\right).$$

Proof. The inflation map $H^1(\Gamma, \mathsf{H}(\mathbb{Q})) \to H^1(\mathbb{Q}, \mathsf{G})$ is injective by [35, Section 5.8.(a)] and lands $Ш^1(\mathbb{Q}, \mathsf{G})$ by the discussion above, and so we are done. □

Since F is totally real, it follows that $\mathsf{H}(\mathbb{R}) = \mathrm{Hom}(\Gamma, \mathsf{G}(\mathbb{R}))$ and therefore by Lemma 2.2.1 that $H^1(\Gamma, \mathsf{H}(\mathbb{R})) = \{1\}$. Thus we observe that

$$\ker\left(H^1(\Gamma, \mathsf{H}(\mathbb{Q})) \to H^1(\Gamma, \mathsf{H}(\mathbb{A}_f))\right) = \ker\left(H^1(\Gamma, \mathsf{H}(\mathbb{Q})) \to H^1(\Gamma, H(\mathbb{A}))\right).$$

Lemma 3.3.2. *If* $Ш^1(\mathbb{Q}, \mathsf{G}) \to Ш^1(\mathsf{F}, G)$ *is injective, then the natural map*

$$\mathsf{G}(\mathbb{Q})\backslash \mathsf{X} \times \mathsf{G}(\mathbb{A}_f) \to \left(\mathsf{H}(\mathbb{Q})\backslash \mathsf{Y} \times \mathsf{H}(\mathbb{A}_f)\right)^{\Gamma}$$

is a bijection.

Proof. By [35, Corollary 1 on page 50], there is a short exact sequence of pointed sets

$$1 \to \mathsf{H}(\mathbb{Q})^{\Gamma} \to \mathsf{H}(\mathbb{A})^{\Gamma} \to \left(\mathsf{H}(\mathbb{Q})\backslash \mathsf{H}(\mathbb{A})\right)^{\Gamma} \to H^1(\Gamma, \mathsf{H}(\mathbb{Q})) \to H^1(\Gamma, \mathsf{H}(\mathbb{A})) \to \cdots$$

It moreover follows from loc. cit. that if $H^1(\Gamma, \mathsf{H}(\mathbb{Q})) \to H^1(\Gamma, \mathsf{H}(\mathbb{A}))$ has trivial kernel then $\mathsf{H}(\mathbb{A})^\Gamma \to (\mathsf{H}(\mathbb{Q})\backslash\mathsf{H}(\mathbb{A}))^\Gamma$ is surjective. Thus by Lemma 3.3.1 and our assumption, the natural map

$$\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}) \to (\mathsf{H}(\mathbb{Q})\backslash\mathsf{H}(\mathbb{A}))^\Gamma$$

is a bijection. Let $\mathsf{K}_\mathsf{X} \subset \mathsf{G}(\mathbb{R})$ be the stabilizer of some point $x \in \mathsf{X} \subset$ and let K_Y be its stabilizer inside of $\mathsf{H}(\mathbb{R})$. Since F is totally real it follows that $\mathsf{H}(\mathbb{R}) = \mathrm{Hom}(\Gamma, \mathsf{G}(\mathbb{R}))$ and also that $\mathsf{K}_\mathsf{Y} = \mathrm{Hom}(\Gamma, \mathsf{K}_\mathsf{X})$; thus $H^1(\Gamma, \mathsf{K}_\mathsf{Y}) = \{1\}$ and $\mathsf{K}_\mathsf{X} = \mathsf{K}_\mathsf{Y}^\Gamma$.

We can identify the natural map of the lemma with

$$\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})/\mathsf{K}_\mathsf{X} \to (\mathsf{H}(\mathbb{Q})\backslash\mathsf{H}(\mathbb{A})/\mathsf{K}_\mathsf{Y})^\Gamma.$$

The lemma now follows from that fact that $H^1(\Gamma, \mathsf{K}_\mathsf{Y}) = \{1\}$ in combination with Corollary A.2.9. ☐

3.4. **Constructing good compact open subgroups.** In order to apply Lemma 3.3.2 to Shimura varieties (or stacks), we need to understand the Γ-cohomology of compact open subgroups of $\mathsf{H}(\mathbb{A}_f)$. In this section, we are going to show that there exist many compact open subgroups with vanishing Γ-cohomology, at least if F is tamely ramified over $\mathbb{Q}$.

Let p be a prime number and write $\mathcal{O}_F := \mathcal{O}_\mathsf{F} \otimes \mathbb{Z}_p$. It is a finite flat algebra over $\mathbb{Z}_p$ equipped with an action of Γ. If $\mathcal{G}$ is a smooth affine group scheme over $\mathbb{Z}_p$ with generic fiber isomorphic to $G := \mathsf{G} \otimes \mathbb{Q}_p$, we define $\mathcal{H} := \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}_p} \mathcal{G}_{\mathcal{O}_F}$. It is a smooth affine group scheme over $\mathbb{Z}_p$ with generic fiber isomorphic to $\mathrm{Res}_{F/\mathbb{Q}_p} G_{F_p}$, where $F = \mathsf{F} \otimes_\mathbb{Q} \mathbb{Q}_p$. Moreover the Galois group Γ acts on $\mathcal{H}$.

Lemma 3.4.1. *As $\mathcal{G}$ runs over all smooth affine group schemes over $\mathbb{Z}_p$ with connected special fiber and with generic fiber isomorphic to G, the groups $\mathcal{H}(\mathbb{Z}_p)$ form a cofinal collection of compact open subgroups of $H(\mathbb{Q}_p)$.*

Proof. By the results [14, Remark A.5.14, Lemma A.5.15] we can construct a sequence of smooth group schemes $\mathcal{G}_1 \leftarrow \mathcal{G}_2 \leftarrow \mathcal{G}_3 \leftarrow \cdots$ over $\mathbb{Z}_p$, such that: The induced maps $\mathcal{G}_{i+1,\mathbb{Q}_p} \to \mathcal{G}_{i,\mathbb{Q}_p}$ are isomorphisms, there is an isomorphism $\mathcal{G}_{1,\mathbb{Q}_p} \simeq G$ and by [14, Lemma A.5.13] we have

$$\bigcap_{i=1}^{\infty} (\mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}_p} \mathcal{G}_{i,\mathcal{O}_F})(\mathbb{Z}_p) = \{1\}.$$

The lemma follows by replacing each $\mathcal{G}_i$ with the open subgroup whose special fiber is the identity component of $\mathcal{G}_{i,\mathbb{F}_p}$. ☐

We now show that if p is unramified in F, then the cohomology of $\mathcal{H}(\mathbb{Z}_p)$ is trivial.

Lemma 3.4.2. *If p is unramified in F, then for each smooth affine group scheme $\mathcal{G}$ over $\mathbb{Z}_p$ with connected special fiber and with generic fiber isomorphic to G, the group $\mathcal{H}(\mathbb{Z}_p) = \mathcal{G}(\mathcal{O}_F)$ satisfies*

$$H^1(\Gamma, \mathcal{H}(\mathbb{Z}_p)) = \{1\}.$$

Proof. Choose a prime $\mathfrak{p}$ of F above p and let $\Gamma' \subset \Gamma$ be its stabilizer. Then there is a Γ-equivariant isomorphism of rings

$$\mathcal{O}_F \simeq \mathrm{Map}_{\Gamma'}(\Gamma, \mathcal{O}_{F,\mathfrak{p}}),$$

which induces a Γ-equivariant isomorphism of groups

$$\begin{aligned}\mathcal{H}(\mathbb{Z}_p) &= \mathcal{G}(\mathcal{O}_F) \\ &= \mathrm{Map}_{\Gamma'}(\Gamma, \mathcal{G}(\mathcal{O}_{F,\mathfrak{p}})).\end{aligned}$$

By Lemma 2.2.1, it then suffices to show that

$$H^1(\Gamma', \mathcal{G}(\mathcal{O}_{F,\mathfrak{p}})) = \{1\}.$$

Consider the inflation map (which is injective by [35, Section 5.8.(a)])

$$H^1(\Gamma', \mathcal{G}(\mathcal{O}_{F,\mathfrak{p}})) \to H^1(\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p, \mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})).$$

Because $\mathcal{H}$ is smooth with connected special fiber, Lang's lemma tells us that

$$H^1(\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p), \mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})) = H^1(\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p), \mathcal{G}(\overline{\mathbb{F}}_p)) = \{1\}$$

and so we are done. □

If p is coprime to the order of Γ, then the cohomology of $\mathcal{H}(\mathbb{Z}_p)$ is trivial if it is a pro-p group.

Lemma 3.4.3. *Let Γ be a finite group acting on a finite p-group M. If the order of Γ is coprime to p, then $H^1(\Gamma, M) = \{1\}$.*

Proof. We prove this by strong induction on k where p^k is the order of M. If $k = 1$ then the result is true because then $M \simeq \mathbb{F}_p$ is abelian and Γ is a group of order prime-to-p.

To prove the induction step, we use the well-known fact that the center $Z \subset M$ of M is nontrivial because M is a p-group. Since Γ acts via group homomorphisms, we see that Z is Γ-stable. We now study the long exact sequence in Γ-cohomology for

$$1 \to Z \to M \to M/Z \to 1. \tag{3.4.1}$$

The induction hypothesis tells us that $H^1(\Gamma, M/Z) = \{1\}$ and because Z is abelian and of p-power order it follows that $H^1(\Gamma, Z) = \{1\}$. The induction step then follows from the long exact sequence in Γ-cohomology for (3.4.1). □

Lemma 3.4.4. *If p is coprime to the order of Γ and $\mathcal{H}(\mathbb{Z}_p)$ is pro-p, then $H^1(\Gamma, \mathcal{H}(\mathbb{Z}_p)) = \{1\}$.*

Proof. There is a Γ-equivariant identification $\mathcal{H}(\mathbb{Z}_p) = \varprojlim_n \mathcal{H}(\mathbb{Z}/p^n\mathbb{Z})$. Let $\sigma : \Gamma \to \mathcal{H}(\mathbb{Z}_p)$ be a cocycle and let P be the (possibly empty) set of elements $h \in \mathcal{H}(\mathbb{Z}_p)$ such that for all $\gamma \in \Gamma$ we have $1 = h \cdot \sigma(\gamma) \cdot \alpha_\gamma(h^{-1})$, where $\alpha_\gamma : \mathcal{H}(\mathbb{Z}_p) \to \mathcal{H}(\mathbb{Z}_p)$ denotes the action of γ. Then $\mathcal{H}(\mathbb{Z}_p)^\Gamma$ acts on P by left multiplication on h and this action is simply transitive if P is nonempty. The lemma is asserting that P is nonempty, which we will now prove.

For a positive integer n, we let P_n be the set of elements $h_n \in \mathcal{H}(\mathbb{Z}/p^n\mathbb{Z})$ such that for all $\gamma \in \Gamma$ we have $1 = h_n \cdot \sigma_n(\gamma) \cdot \alpha_\gamma(h_n^{-1})$, where σ_n is the composition of σ with $\mathcal{H}(\mathbb{Z}_p) \to \mathcal{H}(\mathbb{Z}/p^n\mathbb{Z})$. Then P_n is nonempty by Lemma 3.4.3 and $\mathcal{H}(\mathbb{Z}/p^n\mathbb{Z})^\Gamma$ acts simply transitively on P_n by left multiplication on h_n. There are maps $P_{n+1} \to P_n$ which are $\mathcal{H}(\mathbb{Z}/p^{n+1}\mathbb{Z})^\Gamma$-equivariant via the natural map $\mathcal{H}(\mathbb{Z}/p^{n+1}\mathbb{Z})^\Gamma \to \mathcal{H}(\mathbb{Z}/p^n\mathbb{Z})^\Gamma$, and it follows from the definitions that the natural map

$$P \to \varprojlim_n P_n$$

is a bijection. To show that P is nonempty, it is therefore enough to show that the transition maps $P_{n+1} \to P_n$ are surjective. For this, we consider the Γ-equivariant short exact sequence

$$1 \to Q \to \mathcal{H}(\mathbb{Z}/p^{n+1}\mathbb{Z}) \to \mathcal{H}(\mathbb{Z}/p^n\mathbb{Z}) \to 1$$

defining Q. Note that Q is again a p-group, and thus it follows from Lemma 3.4.3 and the long exact sequence in Γ-cohomology that $\mathcal{H}(\mathbb{Z}/p^{n+1}\mathbb{Z})^\Gamma \to \mathcal{H}(\mathbb{Z}/p^n\mathbb{Z})^\Gamma$ is surjective. We deduce that $P_{n+1} \to P_n$ is surjective, which concludes the proof. □

Lemma 3.4.5. *Suppose that p is tamely ramified in F. Let $\mathcal{G}$ be a smooth affine group scheme over $\mathbb{Z}_p$ with connected special fiber and with generic fiber isomorphic to G. If $\mathcal{H}(\mathbb{Z}_p) = \mathcal{G}(\mathcal{O}_F)$ is a pro-p group then*

$$H^1(\Gamma, \mathcal{H}(\mathbb{Z}_p)) = \{1\}.$$

Proof. Choose a prime $\mathfrak{p}$ of F above p and let L be the completion of F at $\mathfrak{p}$. Let $\mathrm{Gal}(L/\mathbb{Q}_p) \subset \Gamma$ be the stabilizer of $\mathfrak{p}$. As in the proof of Lemma 3.4.2, we can use Lemma 2.2.1 to reduce the lemma can be reduced to proving that

$$H^1(\mathrm{Gal}(L/\mathbb{Q}_p), \mathcal{G}(\mathcal{O}_L)) = \{1\}.$$

Let $L_0 \subset L$ be the maximal unramified subfield of L and consider the short exact sequence of Galois groups

$$1 \to I_L \to \mathrm{Gal}(L/\mathbb{Q}_p) \to \mathrm{Gal}(L_0/\mathbb{Q}_p) \to 1.$$

Then we get an inflation restriction exact sequence of pointed sets

$$\cdots \to H^1(\mathrm{Gal}(L_0/\mathbb{Q}_p), \mathcal{G}(\mathcal{O}_L)^{I_L}) \to H^1(\mathrm{Gal}(L/\mathbb{Q}_p), \mathcal{G}(\mathcal{O}_L)) \to H^1(I_L, \mathcal{G}(\mathcal{O}_L)).$$

The natural map $\mathcal{G}(\mathcal{O}_{L_0}) \to \mathcal{G}(\mathcal{O}_L)^{I_L}$ is an isomorphism and so the first term is trivial by Lemma 3.4.2. The assumption that p is tamely ramified in F means that I_L has order prime to p and thus $H^1(I_L, \mathcal{G}(\mathcal{O}_L))$ vanishes by Lemma 3.4.4 and the fact that $\mathcal{H}(\mathbb{Z}_p)$ is pro-p. It now follows from [35, Corollary 1 on p. 52] that $H^1(\mathrm{Gal}(L/\mathbb{Q}_p), \mathcal{G}(\mathcal{O}_L)) = \{1\}$. □

3.4.6. The results proved above provide many compact open subgroups $K \subset \mathsf{H}(\mathbb{A}_f^p)$ that are Γ-stable and have $H^1(\Gamma, K) = \{1\}$, at least when F is tamely ramified over $\mathbb{Q}$.

Proposition 3.4.7. *Suppose that F is tamely ramified over $\mathbb{Q}$. Then the collection of compact open subgroups $K \subset \mathsf{H}(\mathbb{A}_f)$ that are Γ-stable and satisfy $H^1(\Gamma, K) = \{1\}$ is cofinal in the set of all compact open subgroups of $\mathsf{H}(\mathbb{A}_f)$.*

We will call such compact open subgroups *good*, and we will use the same terminology for Γ-stable compact open subgroups $K \subset G(\mathbb{A}_f^S)$, for some finite set of places S of $\mathbb{Q}$, that satisfy $H^1(\Gamma, K) = \{1\}$.

Proof of Proposition 3.4.7. We can choose groups K of the form

$$K = \prod_p K_p,$$

with $K_p = \mathcal{H}(\mathbb{Z}_p)$ for $H = \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}_p} \mathcal{G}_{\mathcal{O}_F}$ for some smooth affine group scheme $\mathcal{G}$ with connected special fiber and generic fiber G, such that $\mathcal{G}$ is a reductive model of G for all but finitely many p. This collection of compact open subgroups $K \subset \mathsf{H}(\mathbb{A}_f)$ is cofinal by Lemma 3.4.1. We can moreover assume that either p is unramified in F or that K_p is a pro-p group, without affecting co-finality.

Then for primes p unramified in F we have

$$H^1(\Gamma, K_p) = \{1\}$$

by Lemma 3.4.2, and for primes p ramified in F we have $H^1(\Gamma, K_p) = \{1\}$ by Lemma 3.4.5. Thus we find that

$$H^1(\Gamma, K) = \prod_p H^1(\Gamma, K_p) = \{1\}$$

and the result is proved. □

3.5. **Homotopy fixed points of Shimura stacks.** Let the notation be as in the beginning of Section 3. If $K \subset \mathsf{H}(\mathbb{A}_f)$ is a Γ-stable compact open subgroup then there is a natural morphism of groupoids (see Section A.2.6)

$$\left[\mathsf{G}(\mathbb{Q})\backslash \left(\mathsf{X} \times \mathsf{G}(\mathbb{A}_f)/K^{\Gamma}\right)\right] \to \left[\mathsf{H}(\mathbb{Q})\backslash \left(\mathsf{Y} \times \mathsf{H}(\mathbb{A}_f)/K\right)\right]^{h\Gamma}. \tag{3.5.1}$$

In this section we will investigate when this morphism is an equivalence.

Theorem 3.5.1. *If $K \subset \mathsf{H}(\mathbb{A}_f)$ is a good compact open subgroup and if $\text{Ш}^1(\mathbb{Q}, \mathsf{G}) \to \text{Ш}^1(\mathsf{F}, G)$ is injective, then the natural functor*

$$\left[\mathsf{G}(\mathbb{Q})\backslash \left(\mathsf{X} \times \mathsf{G}(\mathbb{A}_f)/K^{\Gamma}\right)\right] \to \left[\mathsf{H}(\mathbb{Q})\backslash \left(\mathsf{Y} \times \mathsf{H}(\mathbb{A}_f)/K\right)\right]^{h\Gamma}$$

is an equivalence of groupoids.

This theorem is close to being optimal. It follows from the discussion in Section 3.1 that the assumption that $\text{Ш}^1(\mathbb{Q}, \mathsf{G}) \to \text{Ш}^1(\mathsf{F}, G)$ is injective is necessary. It follows from Section 3.1 that it is necessary that $H^1(\Gamma, K) \to H^1(\Gamma, \mathsf{H}(\mathbb{A}_f))$ has trivial kernel.

Corollary 3.5.2. *Suppose that the $\mathbb{R}$-split rank of the center of G is zero. If $K \subset \mathsf{H}(\mathbb{A}_f)$ is a neat and good compact open subgroup and $\text{Ш}^1(\mathbb{Q}, \mathsf{G}) \to \text{Ш}^1(\mathsf{F}, G)$ is injective, then the natural morphism of E-varieties*

$$\mathbf{Sh}_{K^{\Gamma}}(\mathsf{G}, \mathsf{X}) \to \mathbf{Sh}_K(\mathsf{H}, \mathsf{Y})^{\Gamma}$$

is an isomorphism.

Proof. It suffices to prove this result after basechanging to $\mathbb{C}$. By [8, Proposition 4.2], the target is smooth because $\mathbf{Sh}_K(\mathsf{H}, \mathsf{Y})$ is smooth. Therefore, it is enough to prove that the map induces a bijection on $\mathbb{C}$-points. Now recall from Lemma 3.2.2 that both (G, X) and (H, Y) satisfy SV5. Since K is neat it follows that $K^{\Gamma} = K \cap \mathsf{G}(\mathbb{A}_f)$ is neat and so by Lemma

3.2.1, the groupoid quotients in the statement of Theorem 3.5.1 are equivalent to the set theoretic quotients. The corollary now follows from Theorem 3.5.1. □

To prove Theorem 3.5.1, it is more convenient to work with a different presentation of the morphisms of groupoids in equation (3.5.1). Informally, we want to swap around the order in which we are taking the quotient.[3]

A compact open subgroup $K \subset \mathsf{H}(\mathbb{A}_f)$ acts on $\mathsf{Y} \times \mathsf{H}(\mathbb{A}_f)$ by $k \cdot (x, g) = (x, g \cdot k)$. This action commutes with the action of $\mathsf{H}(\mathbb{Q})$ and therefore descends to an action of K on

$$\mathsf{H}(\mathbb{Q})\backslash(\mathsf{Y} \times \mathsf{H}(\mathbb{A}_f)).$$

Moreover, if K is Γ-stable then this action is Γ-semilinear and thus induces an action of Γ on the quotient stack of $\mathsf{H}(\mathbb{Q})\backslash(\mathsf{Y} \times \mathsf{H}(\mathbb{A}_f))$ by K. There is moreover an action of K^Γ on $\mathsf{G}(\mathbb{Q})\backslash(\mathsf{X} \times \mathsf{G}(\mathbb{A}_f))$ and the natural map

$$\mathsf{G}(\mathbb{Q})\backslash(\mathsf{X} \times \mathsf{G}(\mathbb{A}_f)) \to \mathsf{H}(\mathbb{Q})\backslash(\mathsf{Y} \times \mathsf{H}(\mathbb{A}_f))$$

induces a Γ-equivariant map of quotient stacks, see Section A.2.6. The following lemma tells us that we can indeed swap around the order in which the quotient is taken.

Lemma 3.5.3. *The natural functor*

$$\left[\mathsf{G}(\mathbb{Q})\backslash \left(\mathsf{X} \times \mathsf{G}(\mathbb{A}_f)/K^\Gamma\right)\right] \to \left[\mathsf{H}(\mathbb{Q})\backslash \left(\mathsf{Y} \times \mathsf{H}(\mathbb{A}_f)/K\right)\right]^{h\Gamma}$$

is an equivalence if and only if

$$\left[\left(\mathsf{G}(\mathbb{Q})\backslash \left(\mathsf{X} \times \mathsf{G}(\mathbb{A}_f)\right)\right)/K^\Gamma\right] \to \left[\left(\mathsf{H}(\mathbb{Q})\backslash \left(\mathsf{Y} \times \mathsf{H}(\mathbb{A}_f)\right)\right)/K\right]^{h\Gamma}$$

is an equivalence.

Proof. There is a Γ-semilinear left action of $\mathsf{H}(\mathbb{Q}) \times K$ on $\mathsf{Y} \times \mathsf{H}(\mathbb{A}_f)$ given by $(h, k) \cdot (x, g) = (h \cdot x, h \cdot g \cdot k^{-1})$ and similarly there is a left action of $\mathsf{G}(\mathbb{Q}) \times K^\Gamma$ on $\mathsf{X} \times \mathsf{G}(\mathbb{A}_f)$. By Lemma A.2.3 there are natural Γ-equivariant functors

$$\begin{aligned} \left[(\mathsf{H}(\mathbb{Q}) \times K)\backslash(\mathsf{Y} \times \mathsf{H}(\mathbb{A}_f))\right] &\to \left[\left(\mathsf{H}(\mathbb{Q})\backslash \left(\mathsf{Y} \times \mathsf{H}(\mathbb{A}_f)\right)\right)/K\right] \\ \left[(\mathsf{H}(\mathbb{Q}) \times K)\backslash(\mathsf{Y} \times \mathsf{H}(\mathbb{A}_f))\right] &\to \left[\mathsf{H}(\mathbb{Q})\backslash \left(\mathsf{Y} \times \mathsf{H}(\mathbb{A}_f)/K\right)\right], \end{aligned}$$

that are easily seen to be equivalences of categories. These morphisms also exist for (G, X) and induce a commutative diagram (see Lemma A.2.3 and equation A.2.1)

$$\begin{array}{ccc} \left[\mathsf{G}(\mathbb{Q})\backslash \left(\mathsf{X} \times \mathsf{G}(\mathbb{A}_f)/K^\Gamma\right)\right] & \longrightarrow & \left[\mathsf{H}(\mathbb{Q})\backslash \left(\mathsf{Y} \times \mathsf{H}(\mathbb{A}_f)/K\right)\right]^{h\Gamma} \\ \uparrow & & \uparrow \\ \left[(\mathsf{G}(\mathbb{Q}) \times K^\Gamma)\backslash(\mathsf{X} \times \mathsf{G}(\mathbb{A}_f))\right] & \longrightarrow & \left[(\mathsf{H}(\mathbb{Q}) \times K)\backslash(\mathsf{Y} \times \mathsf{H}(\mathbb{A}_f))\right]^{h\Gamma} \\ \downarrow & & \downarrow \\ \left[\left(\mathsf{G}(\mathbb{Q})\backslash \left(\mathsf{X} \times \mathsf{G}(\mathbb{A}_f)\right)\right)/K^\Gamma\right] & \longrightarrow & \left[\left(\mathsf{H}(\mathbb{Q})\backslash \left(\mathsf{Y} \times \mathsf{H}(\mathbb{A}_f)\right)\right)/K\right]^{h\Gamma}. \end{array}$$

[3] What follows is presumably a tautology for the readers well versed in 2-category theory. We have opted to spell out the details for our own benefit.

The vertical maps are all equivalences of categories, and thus if one of the horizontal maps is an equivalence, then they all are. □

Proof of Theorem 3.5.1. It follows from the assumptions of the theorem and Lemma 3.3.2 that the natural map

$$(\mathsf{G}(\mathbb{Q})\backslash(\mathsf{X}\times\mathsf{G}(\mathbb{A}_f)) \to (\mathsf{H}(\mathbb{Q})\backslash(\mathsf{Y}\times\mathsf{H}(\mathbb{A}_f)))^{\Gamma}$$

is a bijection. To show that the natural map

$$\left[(\mathsf{G}(\mathbb{Q})\backslash(\mathsf{X}\times\mathsf{G}(\mathbb{A}_f)))/K^{\Gamma}\right] \to \left[(\mathsf{H}(\mathbb{Q})\backslash(\mathsf{Y}\times\mathsf{H}(\mathbb{A}_f)))/K\right]^{h\Gamma} \tag{3.5.2}$$

is an equivalence, it suffices by Corollary A.2.9 to show that $H^1(\Gamma, K) = \{1\}$, which is true by assumption. Therefore, the map in equation (3.5.2) is an equivalence and the theorem now follows from Lemma 3.5.3. □

The following corollary is a direct consequence of the proof of Theorem 3.5.1.

Corollary 3.5.4. *If $K \subset \mathsf{H}(\mathbb{A}_f)$ be a Γ-stable subgroup (not necessarily compact open!) with $H^1(\Gamma, K) = \{1\}$, then the natural map*

$$\left[\mathsf{G}(\mathbb{Q})\backslash\left(\mathsf{X}\times\mathsf{G}(\mathbb{A}_f)/K^{\Gamma}\right)\right] \to \left[\mathsf{H}(\mathbb{Q})\backslash(\mathsf{Y}\times\mathsf{H}(\mathbb{A}_f)/K)\right]^{h\Gamma}$$

is an equivalence. In particular, the right-hand side is equivalent to a discrete groupoid if (G,X) satisfies SV5 and K is neat.

3.6. **Shimura varieties of Hodge type.** Let the notation be as in the beginning of Section 3. For a symplectic space (V, ψ) over $\mathbb{Q}$ we write $\mathsf{G}_V := \mathrm{GSp}(V, \psi)$ for the group of symplectic similitudes of V over $\mathbb{Q}$. It admits a Shimura datum H_V consisting of the union of the Siegel upper and lower half spaces. Assume furthermore that (G,X) is of Hodge type and let $\iota : (\mathsf{G},\mathsf{X}) \to (\mathsf{G}_V, \mathsf{H}_V)$ be a closed immersion of Shimura data.

Recall the following construction from [19, Section 7.1.6][4]. Let W be the symplectic space $V \otimes_{\mathbb{Q}} F$ considered as a vector space over $\mathbb{Q}$, equipped with the symplectic form ψ_W given by

$$W \times W \xrightarrow{\psi\otimes_{\mathbb{Q}} F} F \xrightarrow{\mathrm{Tr}_{\mathsf{F}/\mathbb{Q}}} \mathbb{Q}.$$

Let $c_{\mathsf{G}} : \mathsf{G} \to \mathbb{G}_m$ be the restriction of the symplectic similitude character of G_V to G and let $c_{\mathsf{G},\mathsf{F}} : \mathsf{H} \to \mathrm{Res}_{\mathsf{F}/\mathbb{Q}}\,\mathbb{G}_m$ be the induced map. Form the fiber product

$$\begin{array}{ccc} \mathsf{H}_3 & \longrightarrow & \mathbb{G}_m \\ \downarrow & & \downarrow \\ \mathsf{H} & \xrightarrow{c_{\mathsf{G},\mathsf{F}}} & \mathrm{Res}_{\mathsf{F}/\mathbb{Q}}\,\mathbb{G}_m \end{array}$$

and let H_1 be the neutral component of H_3. Then H_1 is a connected reductive group over $\mathbb{Q}$ and the natural map $\mathsf{G} \to \mathsf{H}$ factors over H_1. For $h : \mathbb{S} \to \mathsf{G}_{\mathbb{R}}$ in X we let Y_1 be the $\mathsf{H}_1(\mathbb{R})$-conjugacy class of the composition $h : \mathbb{S} \to \mathsf{G}_{\mathbb{R}} \to \mathsf{H}_{1,\mathbb{R}}$, this does not depend on the

[4]They use $\mathbf{H}'$ for what we call H and they use $\mathbf{H}$ for what we call H_1.

choice of h. We will write $\mathsf{G}_{V,\mathsf{F}}$ for the inverse image of $\mathbb{G}_m \subset \mathrm{Res}_{\mathsf{F}/\mathbb{Q}} \mathbb{G}_m$ under the natural map

$$\mathrm{Res}_{\mathsf{F}/\mathbb{Q}} G_V \to \mathrm{Res}_{\mathsf{F}/\mathbb{Q}} \mathbb{G}_m.$$

As explained above, this also comes equipped with a Shimura datum $\mathfrak{H}_{V,\mathsf{F}}$. There is moreover a commutative diagram

$$\begin{array}{ccccc} \mathsf{G} & \longrightarrow & \mathsf{H}_1 & & \\ \downarrow{\scriptstyle \iota} & & \downarrow & \searrow^{\iota_{\mathsf{H}_1}} & \\ \mathsf{G}_V & \longrightarrow & \mathsf{G}_{V,\mathsf{F}} & \longrightarrow & \mathsf{G}_W, \end{array}$$

where each map is a closed immersion compatible with the natural Shimura data.

Remark 3.6.1. The group H_1 does not depend on the choice of ι, since the map c_{G} does not depend on the choice of ι, see [4, Lemma 6.1.1].

3.6.2. Shimura varieties of Hodge type automatically satisfy SV5 and in fact their centers have $\mathbb{Q}$-split and $\mathbb{R}$-split ranks equal to 1. Recall from Lemma 3.2.2 that if $[F:\mathbb{Q}] > 1$ then (H,Y) does not satisfy SV5. Since $(\mathsf{H}_1,\mathsf{Y}_1)$ is of Hodge type, it does satisfy SV5. The point of the construction of H_1 is both to remedy the failure of SV5 and to build something that is again of Hodge type.

We would like to compare the Γ-fixed points of Shimura varieties for $(\mathsf{H}_1,\mathsf{Y}_1)$ to the Shimura varieties for (G,X). It seems complicated to do this directly, so we will instead compare the $\mathbb{C}$-points of the Γ-fixed points of the Shimura varieties for $(\mathsf{H}_1,\mathsf{Y}_1)$, to the Γ-homotopy fixed points of the Shimura stacks for (H,Y). We will need the following modification of [16, Lemma 2.1.2], cf. [23, Lemma 2.4.3].

Proposition 3.6.3. *Let ℓ be a prime number coprime to the order of Γ. Let $K^\ell \subset \mathsf{H}(\mathbb{A}_f^\ell)$ be a neat good compact open subgroup and let $K_\ell \subset \mathsf{G}(\mathbb{Q}_\ell)$ be a Γ-stable pro-ℓ group. Define $K = K^\ell K_\ell$ and define $K_{1,\ell} = K_\ell \cap \mathsf{H}_1(\mathbb{Q}_\ell)$ and $K^{1,\ell} = K^\ell \cap \mathsf{H}_1(\mathbb{A}_f^\ell)$. Then the natural map*

$$\left(\mathsf{H}_1(\mathbb{Q})\backslash \mathsf{Y}_1 \times \mathsf{H}_1(\mathbb{A}_f)/K_1^\ell K_{1,\ell}\right)^\Gamma \to \left[\mathsf{H}(\mathbb{Q})\backslash \mathsf{Y} \times \mathsf{H}(\mathbb{A}_f)/K^\ell K_{1,\ell}\right]^{h\Gamma}$$

is fully faithful.[5]

Proof of Proposition 3.6.3. We first prove that the natural functor

$$\left(\mathsf{H}_1(\mathbb{Q})\backslash \mathsf{Y}_1 \times \mathsf{H}_1(\mathbb{A}_f)/K_1^\ell\right)^\Gamma \to \left[\mathsf{H}(\mathbb{Q})\backslash \mathsf{Y} \times \mathsf{H}(\mathbb{A}_f)/K^\ell\right]^{h\Gamma}$$

is fully faithful. For this we consider the commutative diagram

$$\begin{array}{ccc} \mathsf{H}_1(\mathbb{Q})\backslash \mathsf{Y}_1 \times \mathsf{H}_1(\mathbb{A}_f)/K_1^\ell & \longrightarrow & \left[\mathsf{H}(\mathbb{Q})\backslash \mathsf{Y} \times \mathsf{H}(\mathbb{A}_f)/K^\ell\right] \\ & \searrow & \downarrow \\ & & \mathsf{H}(\mathbb{Q})\backslash \mathsf{Y} \times \mathsf{H}(\mathbb{A}_f)/K^\ell. \end{array}$$

[5] The groupoid on the right hand side is equivalent to a discrete groupoid by Corollary 3.5.4, and thus the lemma is equivalent to saying that the map on isomorphism classes of objects is injective.

The proof of [23, Lemma 2.4.3] establishes that the diagonal arrow is injective: Indeed, we consider the commutative diagram

$$\begin{array}{ccc} \mathsf{H}_1(\mathbb{Q})\backslash \mathsf{Y}_1 \times \mathsf{H}_1(\mathbb{A}_f)/K_1^{\ell} & \longrightarrow & \mathsf{H}(\mathbb{Q})\backslash \mathsf{Y} \times \mathsf{H}(\mathbb{A}_f)/K^{\ell} \\ \downarrow & & \downarrow \\ \mathsf{H}_1(\mathbb{Q})\backslash H_1(\mathbb{Q}_\ell) & \longrightarrow & \mathsf{H}(\mathbb{Q})\backslash H(\mathbb{Q}_\ell). \end{array}$$

The bottom arrow is injective since $\mathsf{H}(\mathbb{Q}) \cap H_1(\mathbb{Q}_\ell) = \mathsf{H}_1(\mathbb{Q})$. The fibers of the vertical maps can be identified with $\mathsf{Y}_1 \times \mathsf{H}_1(\mathbb{A}_f^{\ell})/K_1^{\ell}$ and $\mathsf{Y} \times \mathsf{H}(\mathbb{A}_f^{\ell})/K^{\ell}$, respectively, and the natural map between them is injective since $K_1^{\ell} = K^{\ell} \cap \mathsf{H}_1(\mathbb{A}_f^{\ell})$. We conclude that the top arrow is injective, and it follows that the diagonal arrow in the following diagram is injective

$$\begin{array}{ccc} \left(\mathsf{H}_1(\mathbb{Q})\backslash \mathsf{Y}_1 \times \mathsf{H}_1(\mathbb{A}_f)/K_1^{\ell}\right)^{\Gamma} & \longrightarrow & \left[\mathsf{H}(\mathbb{Q})\backslash \mathsf{Y} \times \mathsf{H}(\mathbb{A}_f)/K^{\ell}\right]^{h\Gamma} \\ & \searrow & \downarrow \\ & & \left(\mathsf{H}(\mathbb{Q})\backslash \mathsf{Y} \times \mathsf{H}(\mathbb{A}_f)/K^{\ell}\right)^{\Gamma}. \end{array}$$

Since $H^1(\Gamma, K^{\ell}) = \{1\}$ by assumption, it follows from Corollary 3.5.4 that

$$\left[\mathsf{H}(\mathbb{Q})\backslash \mathsf{Y} \times \mathsf{H}(\mathbb{A}_f)/K^{\ell}\right]^{h\Gamma}$$

is a discrete groupoid. Therefore $\left(\mathsf{H}_1(\mathbb{Q})\backslash \mathsf{Y}_1 \times \mathsf{H}_1(\mathbb{A}_f)/K_1^{\ell}\right)^{\Gamma} \to \left[\mathsf{H}(\mathbb{Q})\backslash \mathsf{Y} \times \mathsf{H}(\mathbb{A}_f)/K^{\ell}\right]^{h\Gamma}$ must be a fully faithful map of setoids. To continue the proof we consider the 2-commutative diagram

$$\begin{array}{ccc} \mathsf{H}_1(\mathbb{Q})\backslash \mathsf{Y}_1 \times \mathsf{H}_1(\mathbb{A}_f)/K_1^{\ell} & \longrightarrow & \left[\mathsf{H}(\mathbb{Q})\backslash \mathsf{Y} \times \mathsf{H}(\mathbb{A}_f)/K^{\ell}\right] \\ \downarrow & & \downarrow \\ \mathsf{H}_1(\mathbb{Q})\backslash \mathsf{Y}_1 \times \mathsf{H}_1(\mathbb{A}_f)/K_1^{\ell}K_{1,\ell} & \longrightarrow & \left[\mathsf{H}(\mathbb{Q})\backslash \mathsf{Y} \times \mathsf{H}(\mathbb{A}_f)/K^{\ell}K_{1,\ell}\right], \end{array}$$

which we note is 2-Cartesian since the vertical arrows are essentially surjective with fibers given by the discrete groupoid associated to $K_{1,\ell}$. By Lemma A.1.11, it follows that the diagram stays Cartesian when applying Γ-homotopy fixed points. Since K_ℓ is pro-ℓ by assumption, it follows that $K_{1,\ell}$ is also pro-ℓ and thus $H^1(\Gamma, K_{1,\ell}) = \{1\}$ by Lemma 3.4.4. Therefore, it follows from Corollary A.2.9 that the left vertical map in the diagram

$$\begin{array}{ccc} \left(\mathsf{H}_1(\mathbb{Q})\backslash \mathsf{Y}_1 \times \mathsf{H}_1(\mathbb{A}_f)/K_1^{\ell}\right)^{\Gamma} & \longrightarrow & \left[\mathsf{H}(\mathbb{Q})\backslash \mathsf{Y} \times \mathsf{H}(\mathbb{A}_f)/K^{\ell}\right]^{h\Gamma} \\ \downarrow & & \downarrow \\ \left(\mathsf{H}_1(\mathbb{Q})\backslash \mathsf{Y}_1 \times \mathsf{H}_1(\mathbb{A}_f)/K_1^{\ell}K_{1,\ell}\right)^{\Gamma} & \longrightarrow & \left[\mathsf{H}(\mathbb{Q})\backslash \mathsf{Y} \times \mathsf{H}(\mathbb{A}_f)/K^{\ell}K_{1,\ell}\right], \end{array}$$

is (essentially) surjective, and fully faithfullness of the top row therefore implies the fully faithfulness of the bottom row. $\square$

Theorem 3.6.4. *Let $K \subset \mathsf{H}(\mathbb{A}_f)$ be a neat and good compact open subgroup. If there is a prime number ℓ coprime to the order of Γ such that $K = K^\ell K_\ell$ with K_ℓ a pro-ℓ group, and if $Ш^1(\mathbb{Q}, \mathsf{G}) \to Ш^1(\mathsf{F}, G)$ is injective, then the natural map*

$$\mathbf{Sh}_{K^\Gamma}(\mathsf{G}, \mathsf{X}) \to \mathbf{Sh}_{K_1}(\mathsf{H}_1, \mathsf{Y}_1)^\Gamma$$

is an isomorphism.

Proof. As in the proof of Corollary 3.5.2, we reduce to showing that the map is a bijection on $\mathbb{C}$-points. This map has the form

$$\mathsf{G}(\mathbb{Q})\backslash \mathsf{X} \times \mathsf{G}(\mathbb{A}_f)/K^\Gamma \to (\mathsf{H}_1(\mathbb{Q})\backslash(\mathsf{Y}_1 \times \mathsf{H}_1(\mathbb{A}_f)/K_1)^\Gamma$$

and we consider the composition

$$\begin{aligned}\mathsf{G}(\mathbb{Q})\backslash \mathsf{X} \times \mathsf{G}(\mathbb{A}_f)/K^\Gamma &\to (\mathsf{H}_1(\mathbb{Q})\backslash(\mathsf{Y}_1 \times \mathsf{H}_1(\mathbb{A}_f)/K_1))^\Gamma \\ &\to \left[\mathsf{H}(\mathbb{Q})\backslash(\mathsf{X} \times \mathsf{H}(\mathbb{A}_f)/K^\ell K_{1,\ell})\right]^{h\Gamma}.\end{aligned}$$

Let ℓ and K_ℓ be as in the statement of the theorem. Since K_ℓ is pro-ℓ by assumption, it follows that $K_{1,\ell}$ is also pro-ℓ and thus $H^1(\Gamma, K_{1,\ell}) = \{1\}$ by Lemma 3.4.4. Therefore it follows from Corollary 3.5.4 that this composition is an equivalence of categories. By Proposition 3.6.3, the map

$$\left(\mathsf{H}_1(\mathbb{Q})\backslash \mathsf{Y}_1 \times \mathsf{H}_1(\mathbb{A}_f)/K_1^\ell K_{1,\ell}\right)^\Gamma \to \left[\mathsf{H}(\mathbb{Q})\backslash \mathsf{Y} \times \mathsf{H}(\mathbb{A}_f)/K^\ell K_{1,\ell}\right]^{h\Gamma}$$

is fully faithful and thus it follows that

$$\mathsf{G}(\mathbb{Q})\backslash \mathsf{X} \times \mathsf{G}(\mathbb{A}_f)/K^\Gamma \to (\mathsf{H}_1(\mathbb{Q})\backslash(\mathsf{Y}_1 \times \mathsf{H}_1(\mathbb{A}_f)/K_1))^\Gamma$$

is an equivalence, since its composition with a fully faithful map is an equivalence. Since an equivalence of categories between discrete categories gives a bijection of the underlying sets, we are done. ☐

Remark 3.6.5. Our original approach to proving this theorem was to directly study the Γ-fixed points of the Shimura variety for $(\mathsf{H}_1, \mathsf{Y}_1)$ via non-abelian cohomology methods, which proved quite difficult. This is why it is useful to consider homotopy fixed points of Shimura stacks, even if one is just interested in studying Shimura varieties of Hodge type.

3.6.6. Finally, we collect a result for use in Section 6. For $K \subset \mathsf{H}(\mathbb{A}_f)$ a Γ-stable compact open subgroup we will consider $K^\Gamma \subset \mathsf{G}(\mathbb{A}_f)$ and $K_1 \subset \mathsf{H}_1(\mathbb{A}_f)$.

Lemma 3.6.7. *If $Ш^1(\mathbb{Q}, \mathsf{G}) \to Ш^1(\mathbb{Q}, \mathsf{H})$ is injective, then the natural map of E-schemes*

$$\kappa : \varprojlim_{K \subset \mathsf{H}(\mathbb{A}_f)} \mathbf{Sh}_{K^\Gamma}(\mathsf{G}, \mathsf{X}) \to \left(\varprojlim_{K \subset \mathsf{H}(\mathbb{A}_f)} \mathbf{Sh}_{K_1}(\mathsf{H}_1, \mathsf{Y}_1)\right)^\Gamma,$$

where the limits runs over Γ-stable compact open subgroups, is an isomorphism.

Proof. First of all taking Γ-fixed points commutes with inverse limits, so we are equivalently trying to show that

$$\kappa' : \varprojlim_{K \subset \mathsf{H}(\mathbb{A}_f)} \mathbf{Sh}_{K^\Gamma}(\mathsf{G}, \mathsf{X}) \to \varprojlim_{K \subset \mathsf{H}(\mathbb{A}_f)} \mathbf{Sh}_{K_1}(\mathsf{H}_1, \mathsf{Y}_1)^\Gamma,$$

is an isomorphism. It suffices to prove this after basechanging to $\mathbb{C}$. To prove it is an isomorphism over $\mathbb{C}$, we first show it induces a bijection on $\mathbb{C}$-points. There we are looking at the map (injective by [7, Variante 1.15.1])

$$\mathsf{G}(\mathbb{Q})\backslash \mathsf{X} \times \mathsf{G}(\mathbb{A}_f) \to \left(\mathsf{H}_1(\mathbb{Q})\backslash \mathsf{Y}_1 \times \mathsf{H}_1(\mathbb{A}_f)\right)^\Gamma.$$

As before we consider the composition

$$\mathsf{G}(\mathbb{Q})\backslash \mathsf{X} \times \mathsf{G}(\mathbb{A}_f) \to \left(\mathsf{H}_1(\mathbb{Q})\backslash \mathsf{Y}_1 \times \mathsf{H}_1(\mathbb{A}_f)\right)^\Gamma \to \left(\mathsf{H}(\mathbb{Q})\backslash \mathsf{Y} \times \mathsf{H}(\mathbb{A}_f)\right)^\Gamma,$$

which is an isomorphism by Lemma 3.3.2. It now suffices to show that

$$\left(\mathsf{H}_1(\mathbb{Q})\backslash \mathsf{Y}_1 \times \mathsf{H}_1(\mathbb{A}_f)\right)^\Gamma \to \left(\mathsf{H}(\mathbb{Q})\backslash \mathsf{Y} \times \mathsf{H}(\mathbb{A}_f)\right)^\Gamma$$

is injective, which can be checked before taking Γ-fixed points. But then the result is well known, see [7, Variante 1.15.1].

To deduce that κ' is an isomorphism, we argue as follows: It follows from the proof of [7, Proposition 1.15] that κ' is a closed immersion, thus it suffices to show the map is surjective on topological spaces. Both the source and target are Jacobson schemes since they are integral over the bottom object of the inverse limit, which is Jacobson; thus $\mathbb{C}$-points are dense in the source and target. Furthermore, the map κ' is integral since it is an inverse limit of integral maps of schemes. Therefore κ' is universally closed, and thus surjective since the image is closed and contains the dense set of $\mathbb{C}$-points. □

4. Integral models of Shimura varieties

In this section we prove some results about integral models of Shimura varieties of Hodge type that we will need, in particular some Γ-equivariance properties for the integral models for $(\mathsf{H}_1, \mathsf{Y}_1)$. In Section 4.1 we prove that the formation of shtukas on Shimura varieties is functorial in $(\mathsf{G}, \mathsf{X}, \mathcal{G})$ in a two-categorical manner, which we use to deduce Γ-equivariance the two-categorical sense. This is transferred to shtukas on integral models in Section 4. In Section 4.3, we prove that certain Γ-equivariant morphisms of Shimura varieties can be upgraded to Γ-equivariant morphisms of the corresponding local model diagrams.

4.1. Shtukas on Shimura varieties.

Let (G, X) be a Shimura datum with reflex field E satisfying Milne's axiom SV5 from [26, p. 64]. Fix a prime p and write $G = \mathsf{G} \otimes \mathbb{Q}_p$. Fix a prime v of E above p and let E be the v-adic completion of E with ring of integers $\mathcal{O}_E$ and residue field k_E. Let μ be the induced $G(\overline{\mathbb{Q}}_p)$-conjugacy class of cocharacters of G coming from the Hodge cocharacter of some $x \in \mathsf{X}$ and the choice of place v.

Let $\mathcal{G}$ be a parahoric model of G over $\mathbb{Z}_p$ and set $\mathcal{G}(\mathbb{Z}_p) := K_p \subset G(\mathbb{Q}_p)$. For $K^p \subset \mathsf{G}(\mathbb{A}_f)$ a neat compact open subgroup we write $K = K^p K_p$ and consider the Shimura variety

$\mathbf{Sh}_K(\mathsf{G},\mathsf{X})$ over E. The group $\mathsf{G}(\mathbb{A}_f^p)$ acts on the inverse limit

$$\mathbf{Sh}_{K_p}(\mathsf{G},\mathsf{X}) := \varprojlim_{K^p} \mathbf{Sh}_{K^pK_p}(\mathsf{G},\mathsf{X}).$$

If it is clear from context, we will omit (G,X) from the notation. By [27, Proposition 4.1.2], there are morphisms

$$\mathbf{Sh}_K(\mathsf{G},\mathsf{X})^{\lozenge} \to \mathrm{Sht}_{\mathcal{G},\mu} \otimes_{\mathrm{Spd}\,\mathcal{O}_E} \mathrm{Spd}\,E$$

that are compatible with changing K^p. The goal of this section is to investigate the functoriality of this construction in the triple $(\mathsf{G},\mathsf{X},\mathcal{G})$.

4.1.1. Consider the category ShTrp whose objects are triples $(\mathsf{G},\mathsf{X},\mathcal{G})$, where (G,X) is a Shimura datum satisfying SV5 and where $\mathcal{G}$ is a parahoric model of G. Morphisms in ShTrp are morphisms $(\mathsf{G},\mathsf{X}) \to (\mathsf{G}',\mathsf{X}')$ that extend (necessarily uniquely) to $\mathcal{G} \to \mathcal{G}'$. For $(\mathsf{G},\mathsf{X},\mathcal{G})$ we will write $K_p = \mathcal{G}(\mathbb{Z}_p)$. If we fix an isomorphism $\mathbb{C} \to \overline{\mathbb{Q}}_p$, then the reflex field E of each Shimura datum has a natural embedding $\mathsf{E} \to \mathbb{C} \to \overline{\mathbb{Q}}_p$ with completion E. For $L \subset \overline{\mathbb{Q}}_p$ a finite extension of $\mathbb{Q}_p$, we will write $\mathrm{ShTrp}_L \subset \mathrm{ShTrp}$ for the full subcategory of triples such that the reflex field is contained in L. There is a strict functor

$$\begin{aligned} \mathbf{Sh} : \mathrm{ShTrp}_L &\to \mathcal{D}_L \\ (\mathsf{G},\mathsf{X},\mathcal{G}) &\mapsto \mathbf{Sh}_{K_p}(\mathsf{G},\mathsf{X})^{\lozenge}_L, \end{aligned}$$

and similarly a strict functor sending

$$\mathbf{Sh}_\infty : (\mathsf{G},\mathsf{X},\mathcal{G}) \mapsto \varprojlim_{U_p} \mathbf{Sh}_{U_p}(\mathsf{G},\mathsf{X})^{\lozenge}_L.$$

We also get a strict functor sending $(\mathsf{G},\mathsf{X},\mathcal{G}) \mapsto \underline{K_p}$ and thus a weak functor $\mathbb{B}$ sending $(\mathsf{G},\mathsf{X},\mathcal{G}) \mapsto \mathbb{B}\underline{K_p}$, see Section A.2.6. The natural map

$$\varprojlim_{U_p} \mathbf{Sh}_{U_p}(\mathsf{G},\mathsf{X})^{\lozenge} \to \mathbf{Sh}_{K_p}(\mathsf{G},\mathsf{X})^{\lozenge}$$

is a $\underline{K}_p$-torsor and corresponds to a weak natural transformation (see Lemma A.2.3)

$$\mathbf{Sh} \to \mathbb{B}.$$

4.1.2. There is a natural functor $\mathrm{ShTrp}_L \to \mathrm{ShtPr}_L$ sending $(\mathsf{G},\mathsf{X},\mathcal{G}) \mapsto (\mathcal{G},\mu)$, where μ is the $G(\overline{\mathbb{Q}}_p)$-conjugacy class of cocharacters of G corresponding to our fixed isomorphism $\mathbb{C} \to \overline{\mathbb{Q}}_p$. Thus we can think of $\mathrm{Sht} : \mathrm{ShtPr}_L \to \mathcal{D}_L$ as a weak functor on ShTrp_L. We let $(\mathbb{B}\underline{K_p})^{\mathrm{dR}}$ denote the sub (pre-)stack of $(\mathbb{B}\underline{K_p})$ consisting of K_p-torsors that are de-Rham in the sense of [27, Definition 2.6.1].

Lemma 4.1.3. *The morphism* $\mathbf{Sh}_{K_p}(\mathsf{G},\mathsf{X})^{\lozenge}_L \to \mathbb{B}\underline{K_p}$ *factors through* $(\mathbb{B}\underline{K_p})^{\mathrm{dR}} \subset (\mathbb{B}\underline{K_p})$.

Proof. This follows from the main results of [22], as explained in the proof of [27, Proposition 4.1.2]. □

Proposition 4.1.4. *There is a weak natural transformation* $\mathbf{Sh} \to \mathrm{Sht}$ *factoring on objects through* $\mathrm{Sht}_{\mathcal{G},\mu,L}$*, which recovers the construction of* [27, Section 2.6] *for each triple* $(\mathsf{G}, \mathsf{X}, \mathcal{G})$.

Proof. We are going to chain together a number of weak functors below. In the rest of this proof we let $S = \mathbf{Sh}_{K_p}(\mathsf{G}, \mathsf{X})_L^{\lozenge}$.

Step 1: There is a functor from $(\mathbb{B}\underline{K_p})(S)$ to the groupoid of exact tensor functors $\mathrm{Rep}_{\mathbb{Z}_p} \mathcal{G} \to \{\mathbb{Z}_p \text{ local systems on } S\}$, see [27, Section 2.6.2]. It is given by sending a $\underline{K_p}$-torsor $\mathcal{P} \to S$ to the tensor functor sending a representation $\rho : \mathcal{G} \to \mathrm{GL}(\Lambda)$, where Λ is a finite free $\mathbb{Z}_p$-module, to the $\mathbb{Z}_p$-local system

$$\mathcal{P} \times^{\rho} \underline{\Lambda}.$$

This is weakly functorial in $f : \mathcal{G} \to \mathcal{G}'$ by using the identifications

$$\left(\mathcal{P} \times^{f} \mathcal{G}'\right) \times^{\rho'} \underline{\Lambda} \to \mathcal{P} \times^{\rho} \underline{\Lambda}$$

of Section A.2. Here $\rho' : \mathcal{G}' \to \mathrm{GL}(\Lambda)$ is a representation whose composition with f gives the representation ρ of $\mathcal{G}$.

Per definition of de-Rham local systems, this equivalence identifies $(\mathbb{B}\underline{K_p})^{\mathrm{dR}}(S)$ with the groupoid of exact tensor functors $\mathrm{Rep}_{\mathbb{Z}_p} \mathcal{G} \to \{\mathbb{Z}_p \text{ de-Rham local systems on } S\}$.

Step 2: Pappas and Rapoport construct in [27, Proposition 2.6.3, Definition 2.6.4] an equivalence of categories

$$\{\mathbb{Z}_p \text{ de-Rham local systems on } S\} \to \{\text{Shtukas on } S\}.$$

If we could show this was a tensor equivalence, then composing with this equivalence would give us a functor from the groupoid of exact tensor functors

$$\mathrm{Rep}_{\mathbb{Z}_p} \mathcal{G} \to \{\mathbb{Z}_p \text{ de-Rham local systems on } S\}$$

to the groupoid of exact tensor functors $\mathrm{Rep}_{\mathbb{Z}_p} \mathcal{G} \to \{\text{Shtukas on } S\}$, which is weakly functorial in $f : \mathcal{G} \to \mathcal{G}'$.

The fact that this is a tensor equivalence is implicit in Pappas-Rapoport, we will give a proof for the sake of completeness. By [21, Theorem 3.9] we know that the functor D^0_{dR} defines a tensor functor which is one half of an equivalence of categories between the category of de-Rham local systems $\mathbb{V}$ on a smooth analytic adic space X and the category of filtered $\mathcal{O}_X$ modules with integrable connection satisfying Griffiths transversality. Thus the association of $\mathbb{V}$ to the $\widehat{\mathbb{Z}_p}$-local system[6] $\widehat{\mathbb{V}}$, the $\mathbb{B}^+_{dR,S}$-lattice $\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Z}}_p} \mathbb{B}^+_{dR,S}$ and the $\mathbb{B}^+_{dR,S}$-lattice $\mathbb{V}_0 := (D_{dR}(\mathbb{V}) \otimes_{\mathcal{O}_X} \mathcal{O}\mathbb{B}^+_{dR,S})^{\nabla=0}$, is compatible with tensor products. In particular for $\mathbb{V}, \mathbb{V}'$ the natural map from $\mathbb{V}_0 \otimes \mathbb{V}'_0 \to (\mathbb{V} \otimes \mathbb{V}')_0$ is an isomorphism of $\mathbb{B}^+_{dR,S}$-lattices inside of $(\mathbb{V} \otimes \mathbb{V}')_0 \otimes \mathbb{B}_{dR,S}$. In particular under the canonical comparison $c : \widehat{\mathbb{V}} \otimes \widehat{\mathbb{V}}' \otimes_{\widehat{\mathbb{Z}}_p} \mathbb{B}_{dR,S} \xrightarrow{\sim} (\mathbb{V} \otimes \mathbb{V}')_0 \otimes_{\mathbb{B}^+_{dR,S}} \mathbb{B}_{dR}$, the relative position of $\mathbb{V}_0 \otimes \mathbb{V}'_0$ and $\widehat{\mathbb{V} \otimes \mathbb{V}'} \otimes \mathbb{B}^+_{dR,S}$ is the same as the relative position of $(\mathbb{V} \otimes \mathbb{V}')_0$ with respect to the latter $\mathbb{B}^+_{dR}$ lattice.

[6]Here we use the notation $\widehat{\mathbb{V}}$ to denote the local system on the pro-etale site of X corresponding to the etale local system $\mathbb{V}$.

Now let $S^\lozenge \to \operatorname{Spd}\mathbb{Q}_p$ be the associated diamond of S. Going to the pro-étale cover $\tilde{S}^\lozenge \to S^\lozenge$ over which the local systems $\mathbb{V}$ and $\mathbb{V}'$ are canonically trivialized, the association $\mathbb{V} \to (\mathcal{V}, \varphi_{\mathcal{V}})$ is obtained by gluing the n-dimensional shtuka $(\mathcal{V}_0, \varphi_0)/X_S$ with no legs obtained from the local system associated to $\widehat{\mathbb{V}}$ as in [33, Theorem 12.3.5] by taking $\widehat{\mathbb{V}}$ to $\widehat{\mathbb{V}} \otimes \mathcal{O}_{X_{S,[0,\infty)}} \xrightarrow{\sim} \mathcal{V}_0$, and modifying along the lattice $\mathbb{V}_0$ at each of the points $\phi^n(S^\sharp)$ for $n \geq 0$. In doing so we obtain a new vector bundle $\mathcal{V}/X_S$ with a meromorphic Frobenius $\varphi_{\mathcal{V}} : \varphi^*(\mathcal{V}) \to \mathcal{V}$ obtained from the Frobenius of $\mathcal{V}_0$, which is meromorphic at the point $S^\sharp$. It is easy to see that this process is compatible with tensor products, since the passage from $\widehat{\mathbb{V}}$ to $(\mathcal{V}_0, \varphi_0)$ is evidently a tensor functor, and the lattice we glue along is the tensor product of the lattices associated to $\mathbb{V}, \mathbb{V}'$.

Step 3: Last, we claim that there is an equivalence of categories between the groupoid of exact tensor functors $\operatorname{Rep}_{\mathbb{Z}_p} \mathcal{G} \to \{\text{shtukas over } S\}$ to $\operatorname{Sht}_{\mathcal{G}}(S)$, which is weakly functorial in $\mathcal{G} \to \mathcal{G}'$. The equivalence is explained in [12, Section 5.1.1], and the weak functoriality can be proved in the same way as in Step 1. □

4.2. **The Pappas–Rapoport axioms.** Before we can state the Pappas–Rapoport axioms we introduce some notation. Given a formal scheme $\mathcal{X} \to \operatorname{Spf}\breve{\mathbb{Z}}_p$ that is formally locally of finite type, and a closed point $x \in \mathcal{X}(\overline{\mathbb{F}}_p)$, we denote by $\widehat{\mathcal{X}}_{/x}$, the *formal neighborhood* of $\mathcal{X}$ at x. There is a similar construction for certain v-sheaves over $\operatorname{Spd}\breve{\mathbb{Z}}_p$, see [27, Section 3.3.1]. We will only use this for the tautological point $x_0 \in \mathcal{M}^{\text{int}}_{\mathcal{G},b,\mu}(\operatorname{Spd}\overline{\mathbb{F}}_p)$.

Pappas and Rapoport conjecture, see [27, Conjecture 4.2.2], that there are flat normal integral models $\mathscr{S}_K(\mathsf{G},\mathsf{X}) \to \operatorname{Spec}\mathcal{O}_E$ of $\mathbf{Sh}_K$ for sufficiently small K^p, together with forgetful maps $\mathscr{S}_{K'^pK_p}(\mathsf{G},\mathsf{X}) \to \mathscr{S}_K$ over $\operatorname{Spec}\mathcal{O}_E$ extending the ones on the generic fiber, such that:

a) The forgetful morphisms are finite étale and $\mathsf{G}(\mathbb{A}_f^p)$ acts on the inverse limit

$$\mathscr{S}_{K_p}(\mathsf{G},\mathsf{X}) := \varprojlim_{K^p} \mathscr{S}_{K^pK_p}(\mathsf{G},\mathsf{X}).$$

Moreover, for every discrete valuation ring R over $\mathcal{O}_E$ with characteristic $(0,p)$ the natural map

$$\mathscr{S}_{K_p}(\mathsf{G},\mathsf{X})(R) \to \mathbf{Sh}_{K_p}(\mathsf{G},\mathsf{X})(R[1/p])$$

is a bijection.

b) For all sufficiently small K^p the map $\mathbf{Sh}_K(\mathsf{G},\mathsf{X})^\lozenge \to \operatorname{Sht}_{\mathcal{G},\mu} \otimes_{\operatorname{Spd}\mathcal{O}_E} \operatorname{Spd} E$ extends to a map

$$\mathscr{S}_K(\mathsf{G},\mathsf{X})^{\lozenge/} \to \operatorname{Sht}_{\mathcal{G},\mu}. \tag{4.2.1}$$

c) For all sufficiently small K^p and every $x \in \mathscr{S}_K(\mathsf{G},\mathsf{X})(\overline{\mathbb{F}}_p)$ with induced $b_x : \operatorname{Spd}\overline{\mathbb{F}}_p \to \operatorname{Sht}_{\mathcal{G},\mu}(\overline{\mathbb{F}}_p)$, there is an isomorphism

$$\Theta_x : \mathcal{M}^{\text{int}}_{\mathcal{G},b_x,\mu,/x_0} \to \left(\widehat{\mathscr{S}_{K,/x}}\right)^\lozenge,$$

such that: The pullback under Θ_x of the $\mathcal{G}$-shtuka over $\mathscr{S}_K^{\lozenge/}$ coming from (4.2.1), is isomorphic to the tautological $\mathcal{G}$-shtuka over $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b_x,\mu,/x_0}$.

4.2.1. We will refer to integral models satisfying their conjecture as integral models satisfying the Pappas–Rapoport axioms or as *integral canonical models*. Pappas and Rapoport prove, see [27, Theorem 4.2.4], that such integral models are unique if they exist. These integral models are moreover functorial in the triple $(\mathsf{G}, \mathsf{X}, \mathcal{G})$, see [5, Corollary 4.0.11]. By [5, Theorem I], integral models satisfying their conjecture exist if (G, X) is of Hodge type.

Let us denote by $\mathrm{ShTrp}_{L,\mathrm{SV5},\mathrm{PR}}$ the category of triples $(\mathsf{G}, \mathsf{X}, \mathcal{G})$ such that $\mathsf{E} \to \mathbb{C} \to \overline{\mathbb{Q}}_p$ factors through L, such that (G, X) satisfies SV5 and such that [27, Conjecture 4.2.2] holds for $(\mathsf{G}, \mathsf{X}, \mathcal{G})$. Then as explained above, there is a strict functor $\mathscr{S}^{\lozenge/} : \mathrm{ShTrp}_{L,\mathrm{SV5},\mathrm{PR}} \to \mathcal{D}^{\circ}_{\mathcal{O}_L}$ sending $(\mathsf{G}, \mathsf{X}, \mathcal{G})$ to $\mathscr{S}_{K_p}(\mathsf{G}, \mathsf{X})^{\lozenge/}_{\mathcal{O}_L}$. We can now state an important corollary to Proposition 4.1.4.

Corollary 4.2.2. *There is a weak natural transformation $\mathscr{S} \to \mathrm{Sht}$ of weak functors $\mathrm{ShTrp}_{L,\mathrm{SV5},\mathrm{PR}} \to \mathcal{D}^{\circ}_{\mathcal{O}_L}$, with underlying 1-morphisms the morphisms*

$$\mathscr{S}^{\lozenge/}_{K_p}(\mathsf{G}, \mathsf{X}) \to \mathrm{Sht}_{\mathcal{G},\mu}$$

guaranteed to exist by axiom b).

Proof. Restricting to the generic fiber gives a fully faithful functor

$$\mathrm{Hom}(\mathscr{S}_{K_p}(\mathsf{G}, \mathsf{X})^{\lozenge/}, \mathrm{Sht}_{\mathcal{G},\mu}) \to \mathrm{Hom}(\mathbf{Sh}_{K_p}(\mathsf{G}, \mathsf{X})^{\lozenge}, \mathrm{Sht}_{\mathcal{G},\mu}),$$

by [27, Corollary 2.7.10]. To turn the collection of 1-morphisms specified in the lemma into a weak functor, we have to specify certain coherent isomorphisms satisfying certain properties, see Definition A.1.1. By the fully faithfulness, it suffices to do this on the generic fiber, where the result is Proposition 4.1.4. □

4.2.3. Now we prove some uniqueness results for the maps in axiom c).

Lemma 4.2.4. *Let $\mathscr{S}_K = \mathscr{S}_K(\mathsf{G}, \mathsf{X})$ be an integral model satisfying axioms a) and b) above and let $x \in \mathscr{S}_K(\overline{\mathbb{F}}_p)$. Let $b_x : \mathrm{Spd}\, \overline{\mathbb{F}}_p \to \mathrm{Sht}_{\mathcal{G},\mu}$ denote the induced $\mathcal{G}$-shtuka and let $(\mathcal{P}, \phi_{\mathcal{P}})$ be the $\mathcal{G}$-shtuka over $\left(\widehat{\mathscr{S}_{K,/x}}\right)^{\lozenge}$ coming from b). Then there is a unique isomorphism*

$$\beta : \mathrm{BL}^{\circ}(\mathcal{P}, \phi_{\mathcal{P}}) \simeq \mathrm{BL}^{\circ} \circ b_x,$$

over $\left(\widehat{\mathscr{S}_{K,/x}}\right)^{\lozenge}$ extending the given isomorphism over the closed point. This determines a morphism

$$\Psi_{b_x} : \left(\widehat{\mathscr{S}_{K,/x}}\right)^{\lozenge} \to \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b_x,\mu},$$

and an isomorphism from $(\mathcal{P}, \phi_{\mathcal{P}})$ to the tautological $\mathcal{G}$-shtuka over $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b_x,\mu}$.

Proof. The uniqueness is a direct consequence of [27, Proposition 4.2.5], and the existence is [27, Proposition 4.7.1]. □

Corollary 4.2.5. *Let the notation be as in the statement of Lemma 4.2.4. If axiom c) is moreover satisfied for $\mathscr{S}_K$, then the morphism Ψ_{b_x} is an isomorphism.*

Proof. Let b_x be as in the statement of Lemma 4.2.4. By axiom c) there is a morphism $b' : \operatorname{Spd} \overline{\mathbb{F}}_p \to \mathrm{Sht}_{\mathcal{G},\mu}$ and an isomorphism

$$\Theta_x^{-1} : \left(\widehat{\mathscr{S}_{K,/x}}\right)^{\Diamond} \to \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b',\mu,/x_0}.$$

Moreover there exists an isomorphism $\alpha : (\mathcal{P}, \phi_{\mathcal{P}}) \to \Theta^*(\mathcal{P}', \phi_{\mathcal{P}'})$, where $(\mathcal{P}', \phi_{\mathcal{P}'})$ is the tautological shtuka over $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b',\mu}$. This restricts to an isomorphism $b' \to b_x$ which induces an isomorphism

$$\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b_x,\mu,/x_0} \to \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b',\mu,/x_0}.$$

By the uniqueness proved in Lemma 4.2.4, it follows that under this isomorphism the map Θ_x^{-1} corresponds to Ψ_{b_x}, which is therefore an isomorphism. □

4.3. **Local model diagrams.** The purpose of this section is to discuss local model diagrams for the integral models of Shimura varieties constructed by Kisin–Zhou. In particular, we want to show that these local model diagrams are Γ-equivariant.

4.3.1. Let (G, X) be a Shimura datum of Hodge type with reflex field E, let $p > 2$ be a prime such that p is coprime to the order of $\pi_1(G^{\mathrm{der}})$ and let $G = \mathsf{G} \otimes \mathbb{Q}_p$ as usual.

Hypothesis 4.3.2. Either G splits over a tamely ramified extension, or there is a Hodge embedding $\iota : (\mathsf{G}, \mathsf{X}) \to (\mathsf{G}_V, \mathsf{H}_V)$ such that the identity component G^{Sp} of $(\mathsf{G}, \mathsf{X}) \cap \mathsf{Sp}_V$ is isomorphic to

$$\prod_{i=1}^{s} \mathrm{Res}_{\mathsf{K}_i/\mathbb{Q}} \mathsf{H}_i^{\mathrm{Sp}},$$

where $\mathsf{K}_1, \cdots, \mathsf{K}_s$ are totally real fields and where $\mathsf{H}_i^{\mathrm{Sp}}$ is a connected reductive group over K_i whose base change to each p-adic place of K_i is tamely ramified.

4.3.3. Let $\mathcal{G} := \mathcal{G}_x$ be the Bruhat–Tits stabilizer group scheme (also called stabilizer quasi-parahoric, see [28, Section 2.2]) associated to a point x in the Bruhat–Tits building of G. Let v be a place of E above p and let E be the v-adic completion of E.

Assume that Hypothesis 4.3.2 holds and let ι be the Hodge embedding guaranteed to exist by that hypothesis. After possibly replacing ι by another Hodge embedding and (V, ψ) by another symplectic space over $\mathbb{Q}$, we may assume that: The Hodge embedding ι is good for $(\mathsf{G}, \mathsf{X}, \mathcal{G})$ with respect to some lattice $\Lambda \subset V \otimes \mathbb{Q}_p$, in the sense of [19, Section 5.1.2] and (G, X) satisfies Hypothesis 4.3.2. Indeed, this follows from [19, Lemma 5.1.3], noting that the new Hodge embedding ι' constructed from ι in the proof of [19, Proposition 3.3.18] still satisfies the condition above in terms of G^{SP}. Then $\mathcal{G}$ embeds into the parahoric group scheme $\mathcal{G}_V$ of G_V that is the stabilizer of the lattice Λ.

4.3.4. Let F be a Galois totally real field extension of $\mathbb{Q}$ unramified at p, let Γ be its Galois group and let $F = \mathsf{F} \otimes \mathbb{Q}_p$. Let $(\mathsf{G}, \mathsf{X}) \to (\mathsf{H}_1, \mathsf{Y}_1) \to (\mathsf{H}, \mathsf{Y})$ and $\iota_{H_1} : (\mathsf{H}_1, \mathsf{Y}_1) \to (\mathsf{G}_W, \mathsf{H}_W)$ be as in Section 3.6, where we recall that $W = V \otimes_{\mathbb{Q}} \mathsf{F}$. Let $\mathcal{H}_1$ denote the Bruhat–Tits stabilizer group scheme associated to the image of x under the morphism of buildings $B(G, \mathbb{Q}_p) \to B(H_1, \mathbb{Q}_p)$, and let $\mathcal{H} = \operatorname{Res}_{\mathcal{O}_F/\mathbb{Z}_p} \mathcal{G}_{\mathcal{O}_F}$ be the Bruhat–Tits stabilizer scheme associated to the image of x in $B(H, \mathbb{Q}_p)$.

Recall from [19, Section 2.4.8] that there is a natural map $\beta : \mathcal{H}_1 \to \mathcal{H}$.

Lemma 4.3.5. *If Hypothesis 4.3.2 holds, then the natural map $\mathcal{H}_1 \to \mathcal{H}$ is a closed immersion.*

Proof. By [19, Proposition 2.4.9], it suffices to show that the centraliser T_1 of a maximal $\breve{\mathbb{Q}}_p$ split torus in H_1 is an R-smooth torus (in the sense of [19, Section 5.1.2]). This is automatic when G is tamely ramified, see [19, Proposition 2.4.6]. Otherwise, we observe that

$$H_1^{\mathrm{Sp}} = \operatorname{Res}_{\mathsf{F}/\mathbb{Q}} \left(\operatorname{Res}_{\mathsf{K}_i/\mathbb{Q}} \mathsf{H}_i^{\mathrm{Sp}} \right)_{\mathsf{F}}$$

and this implies as in the proof of [19, Proposition 5.2.6] that T_1 is R-smooth. $\square$

4.3.6. We recall the following commutative diagram of closed embeddings of Shimura data from Section 3.6

$$\begin{array}{ccccc} (\mathsf{G}, \mathsf{X}) & \longrightarrow & (\mathsf{H}_1, \mathsf{Y}_1) & & \\ \downarrow{\scriptstyle \iota} & & \downarrow & \overset{\iota_{H_1}}{\searrow} & \\ (\mathsf{G}_V, \mathsf{H}_V) & \longrightarrow & (\mathsf{G}_{V,\mathsf{F}}, \mathsf{H}_{V,\mathsf{F}}) & \longrightarrow & (\mathsf{G}_W, \mathsf{H}_W). \end{array}$$

Let $\mathcal{G}_W$ be the parahoric group scheme of G_W that is the stabilizer of the lattice $\Lambda_W = \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_F$ and let $\mathcal{GL}_W$ be the reductive integral model of GL_W corresponding to Λ_W. We will also consider the parahoric model $\mathcal{G}_{V,F}$ of $\mathsf{G}_{V,\mathsf{F}}$ corresponding to $\Lambda \otimes \mathcal{O}_F$. We will write $M_{p,F} = \mathcal{G}_{V,F}(\mathbb{Z}_p)$ and $M_{p,W} = \mathcal{G}_W(\mathbb{Z}_p)$.

Lemma 4.3.7. *The morphism ι_{H_1} is a good Hodge embedding with respect to the lattice $\Lambda_W = \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_F \subset W \otimes \mathbb{Q}_p$.*

Proof. We need to check that the morphism ι_{H_1} satisfies the following three properties

(i) $\iota_{H_1}(H_1)$ contains the scalars.
(ii) ι_{H_1} extends to a closed immersion $\iota_{H_1} : \mathcal{H}_1 \to \mathcal{GL}_W$.
(iii) The natural map of local models $\mathbb{M}_{\mathcal{H}_1,\mu} \to \mathbb{M}_{\mathcal{GL}_W,\mu}$ induced by $\mathcal{H}_1 \to \mathcal{GL}_W$ is a closed immersion.[7]

We know that $\iota_{H_1}(H_1)$ contains the scalars because $\iota(G)$ contains the scalars of G_V and the image of the natural map $G_{V,F} \to G_W$ contains the scalars. Property (ii) is a consequence of Lemma 4.3.5 together with the fact that $\mathcal{H} \to \mathcal{GL}_W$ is a closed immersion because it

[7]This condition is equivalent to condition (3) of [19, Definition 3.3.15] because there is a unique natural map extending the natural map on the generic fiber. Moreover, the local models used by [19] agree with ours because theirs also satisfy the Scholze–Weinstein conjecture (which means that they have the correct associated v-sheaf), see [19, Proposition 3.3.10].

is the restriction of scalars along $\operatorname{Spec}\mathcal{O}_F \to \operatorname{Spec}\mathbb{Z}_p$ of (the base change to $\mathcal{O}_F$ of) the closed immersion $\mathcal{G} \to \mathcal{GL}_V$. Property (iii) follows because under our assumptions we have functorial isomorphism (see Lemma 2.4.3)

$$\mathbb{M}_{\mathcal{GL}_W,\mu} \otimes \mathcal{O}_F \simeq \prod_{i=1}^{d} \mathbb{M}_{\mathcal{GL}_V,\mu} \otimes \mathcal{O}_F$$
$$\mathbb{M}_{\mathcal{H}_1,\mu}\mathcal{O}_F \simeq \mathbb{M}_{\mathcal{H},\mu} \otimes \mathcal{O}_F \simeq \prod_{i=1}^{d} \mathbb{M}_{\mathcal{G},\mu} \otimes \mathcal{O}_F$$

and property (iii) holds for $\iota : \mathcal{G} \to \mathcal{G}_V$ by assumption. □

4.3.8. We let $K_p = \mathcal{H}(\mathbb{Z}_p)$ which gives $K_{p,1} = \mathcal{H}_1(\mathbb{Z}_p)$ and $K_p^\Gamma = \mathcal{G}(\mathbb{Z}_p)$. We recall the construction of $\mathscr{S}_{K_{p,1}}(\mathsf{H}_1, \mathsf{Y}_1)$ and $\mathscr{S}_{K_p^\Gamma}(\mathsf{G}, \mathsf{X})$ from [19, Section 5.1.4], and note that they come with natural maps

$$\iota : \mathscr{S}_{K_p^\Gamma}(\mathsf{G}, \mathsf{X}) \to \mathscr{S}_{M_p}(\mathsf{G}_V, \mathsf{H}_V)$$
$$\iota_{H_1} : \mathscr{S}_{K_{p,1}}(\mathsf{H}_1, \mathsf{Y}_1) \to \mathscr{S}_{M_{p,W}}(\mathsf{G}_W, \mathsf{H}_W),$$

where $M_p = \mathcal{G}_V(\mathbb{Z}_p)$ and $M_{p,W} = \mathcal{G}_W(\mathbb{Z}_p)$. The targets of these morphisms have a moduli interpretation in terms of (weakly) polarized abelian schemes (A, λ) up to prime-to-p isogeny with prime-to-p level structures. By [19, Proposition 5.2.2], the $\mathcal{O}_E$ scheme $\mathscr{S}_{K_p^\Gamma}(\mathsf{G}, \mathsf{X})$ is a flat $\mathsf{G}(\mathbb{A}_f^p)$-equivariant extension of $\mathbf{Sh}_{K_p^\Gamma}(\mathsf{G}, \mathsf{X})_E$. Moreover $\mathscr{S}_{K_p^\Gamma}(\mathsf{G}, \mathsf{X})$ fits in a local model diagram

$$\mathscr{S}_{K_p^\Gamma}(\mathsf{G}, \mathsf{X}) \xleftarrow{\;q\;} \widetilde{\mathscr{S}}_{K_p^\Gamma}(\mathsf{G}, \mathsf{X}) \xrightarrow{\;\pi\;} \mathbb{M}_{\mathcal{G},\mu},$$

where q is a $\mathcal{G}$-torsor and π is pro-smooth of relative dimension $\dim G$. We have a similar local model diagram for $\mathscr{S}_{K_{p,1}}(\mathsf{H}_1, \mathsf{Y}_1)$ given by

$$\mathscr{S}_{K_{p,1}}(\mathsf{H}_1, \mathsf{Y}_1) \xleftarrow{\;q_1\;} \widetilde{\mathscr{S}}_{K_{p,1}}(\mathsf{H}_1, \mathsf{Y}_1) \xrightarrow{\;\pi_1\;} \mathbb{M}_{\mathcal{H},\mu},$$

where q_1 is an $\mathcal{H}_1$-torsor and π_1 is pro-smooth of relative dimension $\dim H_1$.

Remark 4.3.9. The map π has an entirely classical description over the generic fiber, see [15, Proposition 4.2.26]. The torsor of trivializations of the first de-Rham cohomology of the universal abelian variety over $\mathscr{S}_{M_p}(\mathsf{G}_V, \mathsf{H}_V)$ has a natural reduction to a $\mathcal{G}$-bundle $\mathcal{V}_{dR}$. The morphism π is given by assigning to a trivialization ϕ of the first de-Rham cohomology, the pre-image $\phi^{-1}(\mathcal{F}^1)$ of the nontrivial step of the Hodge filtration.

4.3.10. We require the following results on the local model diagram.

Proposition 4.3.11. *(1) There exists a Γ-action on $\widetilde{\mathscr{S}_{K_{p,1}}}(\mathsf{H}_1, \mathsf{Y}_1)$ extending that which fits in a commutative diagram*

$$\begin{array}{ccccccc} & & \widetilde{\mathscr{S}_{K_{p,1}}}(\mathsf{H}_1,\mathsf{Y}_1) & \xrightarrow{\widetilde{\iota}_{H_1}} & \widetilde{\mathscr{S}_{M_{p,F}}}(\mathsf{G}_{V,\mathsf{F}},\mathsf{H}_{V,\mathsf{F}}) & & \\ & \swarrow^{q_1} & & \searrow^{\pi_1} & \vdots^{q_{F,V}} & \searrow^{\pi_{F,V}} & \\ \mathscr{S}_{K_{p,1}}(\mathsf{H}_1,\mathsf{Y}_1) & \xrightarrow{\iota_{H_1}} & \mathscr{S}_{M_{p,F}}(\mathsf{G}_{V,\mathsf{F}},\mathsf{H}_{V,\mathsf{F}}) & \leftarrow\cdots & \mathbb{M}_{\mathcal{H},\mu} & \xrightarrow{\iota_{H_1,p}} & \mathbb{M}_{\mathcal{G}_{V,F},\mu}, \end{array}$$

and a Γ-action on $\widetilde{\mathscr{S}_{M_{p,F}}}(\mathsf{G}_{V,\mathsf{F}}, \mathsf{H}_{V,\mathsf{F}})$ such that $\pi_1, \pi_{F,V}, q_1, q_{F,V}, \iota_{H_1,p}, \iota_{H_1}$ are Γ-equivariant.

Proof. To prove the Proposition it suffices to consider the case $(\mathsf{G}, \mathsf{X}) = (\mathsf{G}_V, \mathsf{H}_V)$, whence $(\mathsf{H}_1, \mathsf{Y}_1) = (\mathsf{G}_{V,\mathsf{F}}, \mathsf{H}_{V,\mathsf{F}})$. Indeed, we already know that the Γ-action on $\mathbf{Sh}_{K_1'}(\mathsf{H}_1, \mathsf{Y}_1)$ extends to $\mathscr{S}_{K_1'}(\mathsf{H}_1, \mathsf{Y}_1)$ by [5, Corollary 4.0.11]. Thus we are only trying to show that the action extends to the $\mathcal{H}_1$-torsor, and this admits a closed immersion into the pullback of $\widetilde{\mathscr{S}_{M_{p,F}}}(\mathsf{G}_{V,\mathsf{F}}, \mathsf{H}_{V,\mathsf{F}}) \to \mathscr{S}_{M_{p,F}}(\mathsf{G}_{V,\mathsf{F}}, \mathsf{H}_{V,\mathsf{F}})$ along the Γ-equivariant natural map ι_{H_1}. Moreover, the Γ-equivariance of the induced maps can be checked on the generic fiber and then over $\mathbb{C}$, where it is clear from uniformization.

We may also assume that $\Lambda_V \subset V$ is a self-dual $\mathbb{Z}_p$-lattice whence we may assume the same for $\Lambda_W \subset V_F$. To define the action of Γ on the torsor $\widetilde{\mathscr{S}_{M_{p,F}}}(\mathsf{G}_{V,\mathsf{F}}, \mathsf{H}_{V,\mathsf{F}})$ we may make use of the moduli problem associated to this torsor. Recall that $\mathscr{S}_{M_{p,F}}(\mathsf{G}_{V,\mathsf{F}}, \mathsf{H}_{V,\mathsf{F}})$ classifies, over S a $\mathbb{Z}_p$-scheme, tuples $(A, \lambda, i, \overline{\eta})$ where:

(1) $h : A \to S$ is an abelian scheme of dimension $[\mathsf{F} : \mathbb{Q}] \cdot \dim(V)/2$ considered up to prime-to-p isogeny.
(2) λ is a $\mathbb{Z}_{(p)}^\times$-multiple of a principal polarization.
(3) $i : \mathcal{O}_{\mathsf{F},(p)} \to \mathrm{End}_S(A)$ is a map of rings such that the Rosati involution associated to λ induces the identity on $\mathcal{O}_F$ and such that i satisfies a determinant condition as in [20, Section 5].
(4) η is a trivialization $\eta : V^p(A) \xrightarrow{\sim} V_F \otimes \mathbb{A}_f^p$ compatible with the $\mathcal{O}_F$-action and with the polarization up to a scalar in $\mathbb{A}_f^{p,\times}$.

The scheme $\widetilde{\mathscr{S}_{M_{p,F}}}(\mathsf{G}_{V,\mathsf{F}}, \mathsf{H}_{V,\mathsf{F}})$ classifies quintuples $(A, \lambda, i, \eta, \tau)$ where (A, λ, i, η) are as above and $\tau : \Lambda_{V,F}^\vee \xrightarrow{\sim} R^1 h_* \Omega^*_{A/S}$ is compatible with the symplectic pairing and $\mathcal{O}_F$ actions on the source and target. The Γ-action on the moduli problem $\widetilde{\mathscr{S}_{M_{p,F}}}(\mathsf{G}_{V,\mathsf{F}}, \mathsf{H}_{V,\mathsf{F}})$ sends a tuple $(A, \lambda, i, \eta, \tau)$ to the tuple $(A, \lambda, i \otimes_{\mathcal{O}_{\mathsf{F},(p)},\gamma^{-1}} \mathcal{O}_{\mathsf{F},(p)}, \eta, \gamma(\tau))$, here $\tau : \Lambda_{V,F}^\vee \otimes \mathcal{O}_S \xrightarrow{\sim} R^1 h_* \Omega^*_{A/S}$ and we define $\gamma(\tau)$ to be the composition $\tau \circ (\iota_\gamma \otimes \mathcal{O}_S)$ where $\iota_\gamma : \Lambda_{V,F}^\vee \otimes_{\mathcal{O}_{\mathsf{F},(p)},\gamma^{-1}} \mathcal{O}_F \xrightarrow{\sim} \Lambda_{V,F}^\vee$ is the natural isomorphism arising from the trivial descent datum on $\Lambda_{V,F} \xrightarrow{\sim} \Lambda \otimes \mathcal{O}_F$. By definition, this lifts the Γ action on the moduli problem $\mathscr{S}_{M_{p,F}}(\mathsf{G}_{V,\mathsf{F}}, \mathsf{H}_{V,\mathsf{F}})$. One is left to check that the map $\pi_{F,V}$ is Γ-equivariant, and that this moduli interpretation for the Γ action agrees with the naive description of the Γ-action on the Hilbert–Siegel modular varieties we consider by uniformization. We check the former, the latter is left to the reader. By flatness and normality of $\widetilde{\mathscr{S}}_{M_{p,F}}(\mathsf{G}_{V,\mathsf{F}}, \mathsf{H}_{V,\mathsf{F}})$ and its local model it suffices to check this over the generic fiber, but here we see that

$$\pi_{F,V,\mathbb{Q}_p}(\gamma(A, \lambda, i, \eta, \tau)) = \iota_\gamma^{-1}\tau^{-1}(\mathcal{F}^1(R^1 h_* \Omega^*_{A/S}))$$

which was exactly our definition of the Γ-action on the local model $\mathbb{M}_{\mathcal{G}_{V,F},\mu,\mathbb{Q}_p}$, see Remark 4.3.9. □

Remark 4.3.12. We had originally hoped to prove the Γ-equivariance in Proposition 4.3.11 using the Γ-equivariance of shtukas and the interpretation of the local model diagram in terms of shtukas, see [27, Section 4.9.1]. This does not seem to be possible however, since the scheme-theoretic local model diagram is not uniquely pinned down by the v-sheaf theoretic local model diagram, as discussed in [27, Section 4.9.1].

5. Fixed points of integral models of Shimura varieties

In this section we prove Theorem 1 from the introduction.

5.1. **Fixed points of integral models of Shimura varieties of Hodge type.** Let (G, X) be a Shimura datum of Hodge type, let F be a totally real Galois extension of $\mathbb{Q}$ with Galois group Γ. Let $(\mathsf{G}, \mathsf{X}) \to (\mathsf{H}_1, \mathsf{Y}_1) \to (\mathsf{H}, \mathsf{Y})$ be as in Section 3.6. We let $\mathcal{G}^\circ$ be a parahoric model of G that is the identity component of a Bruhat–Tits stabilizer group scheme $\mathcal{G}$ corresponding to some point x in the Bruhat–Tits building of $G(\mathbb{Q}_p)$. We let $\mathcal{H}^\circ \subset \mathcal{H}$ be the corresponding objects for H, and we let $K_p = \mathcal{H}(\mathbb{Z}_p)$ and $K_p^\circ = \mathcal{H}^\circ(\mathbb{Z}_p)$. We let $\mathcal{H}_1$ be the Bruhat–Tits stabilizer group scheme for H_1 and we let $\mathcal{H}_1' \subset \mathcal{H}_1$ be the inverse image of $\mathcal{H}^\circ \subset \mathcal{H}$. We let $\mathcal{G}^{\mathrm{ad}}$ be the Bruhat–Tits stabilizer group scheme corresponding to the image of x in the building of G^{ad}, and similarly we define $\mathcal{H}^{\mathrm{ad}}$. We also need the parahoric group schemes $\mathcal{G}^{\mathrm{ad},\circ}$ and $\mathcal{H}^{\mathrm{ad},\circ}$, and we observe that

$$\mathcal{H}^{\mathrm{ad}} \simeq \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}_p} \mathcal{G}^{\mathrm{ad}}_{\mathcal{O}_F}, \qquad \mathcal{H}^{\mathrm{ad},\circ} \simeq \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}_p} \mathcal{G}^{\mathrm{ad},\circ}_{\mathcal{O}_F}$$

We set $K_{1,p} = \mathcal{H}_1(\mathbb{Z}_p) = K_p \cap H_1(\mathbb{Q}_p)$ and $K_{1,p}' = \mathcal{H}_1(\mathbb{Z}_p) = K_p' \cap H_1(\mathbb{Q}_p)$. We also have $K_p^\Gamma = \mathcal{G}(\mathbb{Z}_p)$ and $K_p^{\circ,\Gamma} = \mathcal{G}(\mathbb{Z}_p)$. In what follows we let $K^p \subset \mathsf{H}(\mathbb{A}_f^p)$ denote a neat good compact open subgroup such that there is a prime number $\ell \neq p$ coprime to the order of Γ such that $K^p = K^{p,\ell} K_\ell$ with K_ℓ a pro-ℓ group. For such K^p we have $K_1, K_1', K^\Gamma, K^{\circ,\Gamma}$ defined in the obvious way using K_1^p and $K^{p,\Gamma}$. Unfortunately, the notation just introduced conflicts with that of the introduction, but we will need the Bruhat–Tits stabilizer group schemes in the proof of part (2) of Theorem 5.1.2 below.

Note that if F is tamely ramified over $\mathbb{Q}$, then such K^p form a cofinal collection of compact open subgroups by Proposition 3.4.7.

5.1.1. By [5, Corollary 4.1.3] there is an integral model $\mathscr{S}_{K_1'}(\mathsf{G}, \mathsf{X})$ equipped with a shtuka (in the sense of [5, Section 3.1.2])

$$\mathscr{S}_{K_1'}(\mathsf{H}_1, \mathsf{Y}_1)^{\diamondsuit/} \to \mathrm{Sht}_{\mathcal{H}_1,\mu}.$$

We note that these integral models agree, by construction, with the ones of [19] that we used in Section 4.3. We have similar integral models for $K, K^\circ, K_1, K_1', K^\Gamma, K^{\circ,\Gamma}$.

Theorem 5.1.2. *Assume that p is unramified in F, that $p > 2$ and that $Ш^1(\mathbb{Q}, \mathsf{G}) \to Ш^1(\mathsf{F}, G)$.*

(1) The natural map

$$\mathscr{S}_{K^{\circ,\Gamma}}(\mathsf{G},\mathsf{X}) \to \mathscr{S}_{K_1'}(\mathsf{H}_1,\mathsf{Y}_1)^\Gamma. \tag{5.1.1}$$

is a universal homeomorphism.

(2) If p is coprime to $|\Gamma|\cdot|\pi_1(G^{\mathrm{der}})|$ and (G,X) satisfies Hypothesis 4.3.2, then the natural map of (5.1.1) is an isomorphism.

(3) If K_p is hyperspecial, then the natural map of (5.1.1) is an isomorphism.

Proof. **Part (1):** We first observe that the natural map $\mathcal{H}_1' \to \mathcal{H}^{\mathrm{ad}}$ factors through $\mathcal{H}^{\mathrm{ad},\circ}$ because $\mathcal{H}_1' \to \mathcal{H}^{\mathrm{ad}}$ factors through $\mathcal{H}$, and $\mathcal{H}_1'$ maps to $\mathcal{H}^\circ$. Now consider the corresponding map $\mathscr{S}_{K_1'}(\mathsf{H}_1,\mathsf{Y}_1)^{\lozenge/} \to \mathrm{Sht}_{\mathcal{H}_1',\mu} \to \mathrm{Sht}_{\mathcal{H}^{\mathrm{ad},\circ},\mu}$; we claim that it is Γ-equivariant in the 2-categorical sense. It suffices to prove this on the generic fiber as in the proof of Corollary 4.2.2. But then the map factors through a Γ-equivariant map to Shimura varieties for $(\mathsf{G}^{\mathrm{ad}},\mathsf{X}^{\mathrm{ad}})$, and we may apply 4.1.4 to $(\mathsf{G}^{\mathrm{ad}},\mathsf{X}^{\mathrm{ad}})$ to conclude. It thus follows from Lemma 2.6.6, using the fact that p is unramified in F, that there is an isomorphism

$$\mathrm{Sht}_{\mathcal{G}^{\mathrm{ad},\circ},\mu} \to \left(\mathrm{Sht}_{\mathcal{H}^{\mathrm{ad},\circ},\mu}\right)^{h\Gamma}.$$

This induces a map

$$\mathscr{S}_{K_1'}(\mathsf{H}_1,\mathsf{Y}_1)^{\lozenge/,\Gamma} \to \mathrm{Sht}_{\mathcal{G}^{\mathrm{ad},\circ},\mu}$$

compatible with the maps $\mathscr{S}_{K^{\circ,\Gamma}}(\mathsf{G},\mathsf{X})^{\lozenge/} \to \mathrm{Sht}_{\mathcal{G}^\circ,\mu} \to \mathrm{Sht}_{\mathcal{G}^{\mathrm{ad},\circ},\mu}$ and $\mathscr{S}_{K^{\circ,\Gamma}}(\mathsf{G},\mathsf{X})^{\lozenge/} \to \mathscr{S}_{K_1'}(\mathsf{H}_1,\mathsf{Y}_1)^{\lozenge/,\Gamma}$. For $x \in \mathscr{S}_{K_1'}(\mathsf{H}_1,\mathsf{Y}_1)^\Gamma(\overline{\mathbb{F}}_p)$ we let $b_x : \mathrm{Spd}\,\overline{\mathbb{F}}_p \to \mathrm{Sht}_{\mathcal{G}^{\mathrm{ad},\circ},\mu}$ be the corresponding $\mathcal{G}^{\mathrm{ad},\circ}$-shtuka. Then by the proof of Lemma 4.2.4, there is a morphism

$$\Psi_{b_x} : \left(\widehat{\mathscr{S}_{K_1'}}(\mathsf{H}_1,\mathsf{Y}_1)^\Gamma\right)^{\lozenge}_{/x} \to \mathcal{M}^{\mathrm{int}}_{\mathcal{G}^{\mathrm{ad},\circ},b_x,\mu,/x_0},$$

and an isomorphism from the pullback under Ψ_{b_x} of the tautological $\mathcal{G}^{\mathrm{ad},\circ}$-shtuka on $\mathcal{M}^{\mathrm{int}}_{\mathcal{G}^{\mathrm{ad},\circ},b_x,\mu,/x_0}$ to the universal $\mathcal{G}^{\mathrm{ad},\circ}$-shtuka pulled back from $\mathscr{S}_{K_1'}(\mathsf{H}_1,\mathsf{Y}_1)^{\Gamma,\lozenge/}$. If we let d_x denote the induced map $\mathrm{Spd}\,\overline{\mathbb{F}}_p \to \mathrm{Sht}_{\mathcal{H}^{\mathrm{ad},\circ},\mu}$, then by the uniqueness proved in Lemma 4.2.4, there is a commutative diagram

$$\begin{array}{ccc} \left(\widehat{\mathscr{S}_{K_1'}}(\mathsf{H}_1,\mathsf{Y}_1)^\Gamma\right)^{\lozenge}_{/x} & \xrightarrow{\Psi_{b_x}} & \mathcal{M}^{\mathrm{int}}_{\mathcal{G}^{\mathrm{ad},\circ},b_x,\mu,/x_0} \\ \downarrow & & \downarrow \\ \left(\widehat{\mathscr{S}_{K_1'}}(\mathsf{H}_1,\mathsf{Y}_1)\right)^{\lozenge}_{/x} & \xrightarrow{\Psi_{d_x}} & \mathcal{M}^{\mathrm{int}}_{\mathcal{H}^{\mathrm{ad},\circ},b_x,\mu,/x_0}, \end{array}$$

where the vertical arrows are the natural closed immersions. We note that the map Ψ_{d_x} is an isomorphism by Corollary 4.2.5 and the fact that

$$\mathcal{M}^{\mathrm{int}}_{\mathcal{H}_1',b_x,\mu,/x_0} \to \mathcal{M}^{\mathrm{int}}_{\mathcal{H}^{\mathrm{ad},\circ},b_x,\mu,/x_0}$$

is an isomorphism, see [28, Theorem 5.1.2]. It is moreover a direct consequence of the uniqueness proved in Lemma 4.2.4 that the morphism Ψ_{d_x} is Γ-equivariant. Since taking Γ-invariants commutes with taking diamond functors and formal completions, we see using Proposition 2.6.10 that Ψ_{b_x} is the Γ-fixed points of Ψ_{d_x}. In particular, Ψ_{b_x} is an isomorphism.

By [28, Theorem 2.5.4, Theorem 2.5.5] in combination with [10, Corollary 1.4], using our assumption that $p > 2$, there is a flat normal formal scheme $\mathcal{N}$ such that

$$\mathcal{N}^{\lozenge} \simeq \mathcal{M}^{\mathrm{int}}_{\mathcal{G}^{\mathrm{ad},\circ},b_x,\mu,/x_0}.$$

In particular, the unique $\overline{\mathbb{F}}_p$-point of $\mathcal{M}^{\mathrm{int}}_{\mathcal{G}^{\mathrm{ad},\circ},b_x,\mu,/x_0}$ (and thus of $\left(\widehat{\mathscr{S}_{K_1'}}(\mathsf{H}_1,\mathsf{Y}_1)^{\Gamma}\right)_{/x}$) lifts to a $\operatorname{Spf} L$ point for a finite extension L of $\breve{\mathbb{Z}}_p$. Since this is true for all x, this implies that $\mathscr{S}_{K_1'}(\mathsf{H}_1,\mathsf{Y}_1)^{\Gamma}$ is topologically flat over $\mathcal{O}_E$, i.e., that its generic fiber is dense. Let $\mathscr{S}_{K_1'}(\mathsf{H}_1,\mathsf{Y}_1)^{\Gamma,\mathrm{awn}} \to \mathscr{S}_{K_1'}(\mathsf{H}_1,\mathsf{Y}_1)^{\Gamma}$ be the absolute weak normalization, see [36, Lemma 0EUS]; this is a universal homeomorphism and so $\mathscr{S}_{K_1'}(\mathsf{H}_1,\mathsf{Y}_1)^{\Gamma,\mathrm{awn}}$ is again topologically flat over $\mathcal{O}_E$. Then by [1, Lemma 2.13], the natural map

$$\left(\widehat{\mathscr{S}_{K_1'}}(\mathsf{H}_1,\mathsf{Y}_1)^{\Gamma,\mathrm{awn}}\right)^{\lozenge}_{/x} \to \left(\widehat{\mathscr{S}_{K_1'}}(\mathsf{H}_1,\mathsf{Y}_1)^{\Gamma}\right)^{\lozenge}_{/x}$$

is an isomorphism. By the fully faithfulness of the $\lozenge$ functor on absolute weakly normal formal schemes flat separated and topologically of finite type over $\mathbb{Z}_p$, see [1, Theorem 2.16], we find that

$$\left(\widehat{\mathscr{S}_{K_1'}}(\mathsf{H}_1,\mathsf{Y}_1)^{\Gamma,\mathrm{awn}}\right)_{/x} \simeq \mathcal{N}.$$

This implies that the complete local rings of $\mathscr{S}_{K_1'}(\mathsf{H}_1,\mathsf{Y}_1)^{\Gamma,\mathrm{awn}}$ at $\overline{\mathbb{F}}_p$ points are normal, and thus that the local rings are normal by [36, Lemma 0FIZ]. This shows that $\mathscr{S}_{K_1'}(\mathsf{H}_1,\mathsf{Y}_1)^{\Gamma,\mathrm{awn}}$ is normal, because normality (of a quasicompact scheme) can be checked at closed points. By the universal property of the absolute weak normalization, the natural map $\mathscr{S}_{K^\circ,\Gamma}(\mathsf{G},\mathsf{X}) \to \mathscr{S}_{K_1'}(\mathsf{H}_1,\mathsf{Y}_1)^{\Gamma}$ lifts to a finite map

$$\mathscr{S}_{K^\circ,\Gamma}(\mathsf{G},\mathsf{X}) \to \mathscr{S}_{K_1'}(\mathsf{H}_1,\mathsf{Y}_1)^{\Gamma,\mathrm{awn}}.$$

This map is an isomorphism on the generic fiber by Theorem 3.6.4 and thus an isomorphism since the target is normal and source and target are flat over $\mathbb{Z}_p$, see [36, Lemma 0AB1]. The theorem now follows from the fact that $\mathscr{S}_{K_1'}(\mathsf{H}_1,\mathsf{Y}_1)^{\Gamma,\mathrm{awn}} \to \mathscr{S}_{K_1}(\mathsf{H}_1,\mathsf{Y}_1)^{\Gamma}$ is a universal homeomorphism.

Part (2): We assume from now on that p is coprime to the order of Γ and to the order of $\pi_1(G^{\mathrm{der}})$, and that $(\mathsf{G},\mathsf{X},\mathcal{G})$ satisfies Assumption 4.3.2. It then follows from Proposition

4.3.11 that there is a Γ-equivariant and smooth map

$$\mathscr{S}_{K_1}(\mathsf{H}_1, \mathsf{Y}_1) \to \left[\mathbb{M}_{\mathcal{H}^{\mathrm{ad}},\mu}/\mathcal{H}^{\mathrm{ad}}\right].$$

It follows from [8, Proposition 4.2], using the fact that Γ is of order prime-to-p, that the induced map

$$\mathscr{S}_{K_1}(\mathsf{H}_1, \mathsf{Y}_1)^{\Gamma} \to \left[\mathbb{M}_{\mathcal{H}^{\mathrm{ad}},\mu}/\mathcal{H}^{\mathrm{ad}}\right]^{h\Gamma}.$$

is also smooth. Recall that Corollary 2.4.8 tells us that the natural map

$$\left[\mathbb{M}_{\mathcal{G}^{\mathrm{ad}},\mu}/\mathcal{G}^{\mathrm{ad}}\right] \to \left[\mathbb{M}_{\mathcal{H}^{\mathrm{ad}},\mu}/\mathcal{H}^{\mathrm{ad}}\right]^{h\Gamma}$$

is an isomorphism. Thus $\mathscr{S}_{K_1}(\mathsf{H}_1, \mathsf{Y}_1)^{\Gamma}$ has a smooth local model diagram to $\mathbb{M}_{\mathcal{G}^{\mathrm{ad}},\mu}$ and is thus flat over $\mathbb{Z}_p$ and normal since $\mathbb{M}_{\mathcal{G}^{\mathrm{ad}},\mu}$ is, see [10, Corollary 1.4]. The natural map $\mathscr{S}_{K_1'}(\mathsf{H}_1, \mathsf{Y}_1) \to \mathscr{S}_{K_1}(\mathsf{H}_1, \mathsf{Y}_1)$ is Γ-equivariant (by [5, Corollary 4.0.11]) and finite étale (by the proof of [5, Theorem 4.0.13]) and thus the induced map on fixed points

$$\mathscr{S}_{K_1'}(\mathsf{H}_1, \mathsf{Y}_1)^{\Gamma} \to \mathscr{S}_{K_1}(\mathsf{H}_1, \mathsf{Y}_1)^{\Gamma}$$

is also finite étale. It follows that $\mathscr{S}_{K_1'}(\mathsf{H}_1, \mathsf{Y}_1)^{\Gamma}$ is flat over $\mathbb{Z}_p$ and normal since $\mathscr{S}_{K_1}(\mathsf{H}_1, \mathsf{Y}_1)^{\Gamma}$ is. Now the natural map of the theorem is a finite morphism that induces an isomorphism on generic fibers by Theorem 3.6.4. Since the target is flat over $\mathbb{Z}_p$ and normal, it follows from [36, Lemma 0AB1] that the map is an isomorphism.

Part (3): We have a diagram

$$\mathscr{S}_{K^{\circ,\Gamma}}(\mathsf{G}, \mathsf{X}) \to \mathscr{S}_{K_1'}(\mathsf{H}_1, \mathsf{Y}_1)^{\Gamma} \to \mathscr{S}_{K_1'}(\mathsf{H}_1, \mathsf{Y}_1).$$

We first observe that the composite is a closed immersion by [38, Theorem 1.1.1]. We now study the map on complete local rings, let us write

$$\widehat{U}_x := \widehat{\mathscr{S}_{K_1'}}(\mathsf{H}_1, \mathsf{Y}_1)_{/x}, \widehat{V}_y := \widehat{\mathscr{S}_{M_{p,F}}}(\mathsf{G}_{V,\mathsf{F}}, \mathsf{H}_{V,\mathsf{F}})_{/y}$$

for $x \in \widehat{\mathscr{S}_{K_1'}}(\overline{\mathbb{F}}_p)$ and $y \in \mathscr{S}_{M_{p,F}}(\mathsf{G}_{V,\mathsf{F}}, \mathsf{H}_{V,\mathsf{F}})(\overline{\mathbb{F}}_p)$. Similarly for $x \in \mathscr{S}_{K_1'}(\mathsf{H}_1, \mathsf{Y}_1)^{\Gamma}(\overline{\mathbb{F}}_p)$ we write $\widehat{W}_{/x} := \widehat{\mathscr{S}_{K_1'}}(\mathsf{H}_1, \mathsf{Y}_1)^{\Gamma}_{/x}$, and we let x_0 be the corresponding $\overline{\mathbb{F}}_p$ point of $\mathscr{S}_{K^{\circ,\Gamma}}(\mathsf{G}, \mathsf{X})$ (which exists by part (1)) and $\widehat{T}_{x_0}$ the corresponding completion. There is a chain of maps

$$\widehat{T}_{x_0} \to \widehat{W}_{/x} \to \widehat{U}_x \to \widehat{V}_{/y} \to \widehat{\mathbb{M}}_{\mathcal{G}_{V,F},\mu,/\,\mathrm{GM}(y)},$$

where all maps in this chain are closed immersions, and the final map is an isomorphism to the local model for $(\mathcal{G}_{V,F}, \mu)$, see [15, Proposition 4.2.2]. We wish to show that the final map $\theta : \widehat{V}_y \to \widehat{\mathbb{M}}_{\mathcal{G}_{V,F},\mu,/\,\mathrm{GM}(y)}$ induces a Γ-equivariant map on tangent spaces. To do this we use that this map has an explicit moduli theoretic interpretation on schemes that admit divided powers. Let (R_y, m_y) be the complete local ring associated to the special fiber of $\widehat{V}_y$, and let $A_y = R_y/m_y^p$. Then R_y/m_y^p admits trivially a set of divided powers, and also caries an action of the group Γ. If we let (B, m_B) denote the special fiber of the complete local ring associated to $\widehat{\mathbb{M}}_{\mathcal{G}_{V,F},\mu,/\,\mathrm{GM}(y)}$, then the isomorphism $\bar{\theta} : B/m_B^p \to A_y/m_y^p$ has a classical interpretation due to Grothendieck and Messing [24]: A_y/m_y^p is a quotient of the universal deformation ring of the universal principally polarized abelian variety $\mathcal{A}$ with $\mathcal{O}_{F,p}$ action ι_A living over A_y, and the map $\bar{\theta}$ is induced by the map sending $\mathcal{A}$ to the image of the Hodge filtration of $H^1_{dR}(\mathcal{A}/\operatorname{Spec}(A_y/m_y^p))$ inside of the crystalline cohomology $H^1_{cris}(\mathcal{A}_y)_{\operatorname{Spec}(A_y/m_y^p)}$, under the $\mathcal{O}_{F,p}$-equivariant identification $H^1_{dR}(\mathcal{A}/\operatorname{Spec}(A_y/m_y^p)) \xrightarrow{\sim} H^1_{cris}(\mathcal{A}_y)_{\operatorname{Spec}(A_y/m_y^p)}$ (coming from the divided powers and the crystal property of crystalline cohomology). Here we have an action of $\gamma \in \Gamma$ which sends $A_y \to \gamma^*\mathcal{A}_y \xrightarrow{\sim} \mathcal{A}_y$ and thus induces an $\mathcal{O}_{F,p}$-semilinear and A_y/m_y^p linear automorphism of $H^1_{cris}(\mathcal{A}_y)_{\operatorname{Spec}(A_y/m_y^p)}$. Because the map $H^1_{dR}(\mathcal{A}/\operatorname{Spec}(A_y/m_y^p)) \xrightarrow{\sim} H^1_{cris}(\mathcal{A}_y)_{\operatorname{Spec}(A_y/m_y^p)}$ is $\mathcal{O}_{F,p}$-equivariant, it is Γ-equivariant, and as there is a canonical isomorphism $\gamma^*(H^1_{dR}(\mathcal{A}/\operatorname{Spec}(A_y/m_y^p))) \xrightarrow{\sim} H^1_{dR}(\gamma^*\mathcal{A}/\operatorname{Spec}(A_y/m_y^p))$ as $\mathcal{O}_{F,p} \otimes \operatorname{Spec}(A_y/m_y^p))$-modules, we see that the map $\bar{\theta}$ is Γ-equivariant.

Finally, we have a closed immersion $\widehat{T}_{x_0} \to \widehat{W}_x$ where the source is smooth, and we wish to show that this is an isomorphism on tangent spaces. Using [15, Proposition 4.22] we may choose a point $\mathrm{GM}(x) \in \widehat{\mathbb{M}}_{\mathcal{H},\mu}$ with image $\mathrm{GM}(y) \in \widehat{\mathbb{M}}_{\mathcal{G}_{V,F},\mu}$ such that the composition $\widehat{U}_{/x} \to \widehat{V}_y \to \widehat{\mathbb{M}}_{\mathcal{G}_{V,F},\mu,/\,\mathrm{GM}(y)}$ factors through an isomorphism from $\widehat{U}_{/x}$ to $\widehat{\mathbb{M}}_{\mathcal{H},\mu}$. Thus we have a commutative diagram

$$\begin{array}{ccc} \widehat{U}_{/x} & \longrightarrow & \widehat{\mathbb{M}}_{\mathcal{H},\mu,/\,\mathrm{GM}(x)} \\ \downarrow & & \downarrow i^{\mathrm{loc}}_{H_1} \\ \widehat{V}_y & \longrightarrow & \widehat{\mathbb{M}}_{\mathcal{G}_{V,F},\mu,/\,\mathrm{GM}(y)}. \end{array}$$

As the vertical maps in the diagram are Γ-equivariant by Proposition 4.3.11, and the bottom horizontal map is Γ-equivariant on tangent spaces by the result of the previous paragraph, we see that the induced isomorphism $T_x(\widehat{U}_x) \xrightarrow{\sim} T_{GM(x)}(\widehat{\mathbb{M}}_{\mathcal{H},\mu,GM(x)})$ is Γ-equivariant on tangent spaces. Finally we deduce from Proposition 2.4.7 that $T_x(\widehat{W}_{x_0}) \simeq T_x(\widehat{U}_x)^\Gamma \simeq T_{GM(x)}(\widehat{\mathbb{M}}_{\mathcal{H},\mu,GM(x)})^\Gamma \simeq T_{GM(x)}(\widehat{\mathbb{M}}_{\mathcal{G},\mu,GM(x)})$. Thus the map $\widehat{T}_{x_0} \to \widehat{W}_x$ is a closed immersion of local formal schemes with smooth source, which induces an isomorphism on tangent spaces and a universal homeomorphism on topological spaces by part (1). By Lemma 5.1.3 below, it is thus an isomorphism.

□

Lemma 5.1.3. *Let $f : A \to B$ be a surjective local homomorphism of complete local Noetherian $\breve{\mathbb{Z}}_p$-algebras with B flat and formally smooth over $\breve{\mathbb{Z}}_p$. Suppose that f induces a universal homeomorphism on spectra. If f induces an isomorphism on tangent spaces, then it is an isomorphism.*

Proof. The proof is no doubt well known to experts, but we were unable to find a suitable reference. Because B is formally smooth over $\breve{\mathbb{Z}}_p$ we have that it is a complete local ring of the form $\breve{\mathbb{Z}}_p[[X_1, \ldots, X_n]]$. The map $f : A \to B$ admits a section $g : B \to A$, which we may construct by taking $g(X_i)$ to be any $Y \in A$ such that $f(Y) = X_i$. The maximal ideal

m_A is a finitely generated A module as A is Noetherian, and the map $g \circ f|_{m_A} : m_A \to m_A$ is a surjective modulo the maximal ideal m_A by assumption, whence it is a surjection by Nakayama's lemma. Thus we have that f induces an isomorphism $m_B \cong m_A$, and so as m_B contains no nilpotents, m_A contains no nilpotents as well and thus A is reduced. Thus f is an isomorphism. $\square$

Remark 5.1.4. One can prove a version of Part (1) of Theorem 5.1.2 for integral models of Shimura varieties of abelian type (G, X), using Theorem 3.5.1 as input on the generic fiber (thus we require the assumption that the $\mathbb{R}$-split rank of Z_{G} is zero). To do this, one needs integral models satisfying the Pappas–Rapoport axioms, see [6] for this.[8] In fact, the proof of such a theorem is easier than the proof of Theorem 5.1.2, because one can directly work with (H, Y) instead of $(\mathsf{H}_1, \mathsf{Y}_1)$.

Remark 5.1.5. If p is completely split in F, then the proof of part (1) of Theorem 5.1.2 can be adapted to show that the natural map is actually an isomorphism. The key point is that if p is completely split, then one can prove a version of Lemma 2.4.3 for the integral local Shimura varieties for $\mathcal{H}^{\mathrm{ad},\circ}$, which then implies that Proposition 2.6.10 holds on the level of formal schemes. We leave the details to the interested reader.

6. Fixed points of Igusa stacks

In this section we discuss the fixed points of the Igusa stacks of [4]. We use this to prove a version of Theorem 5.1.2 only assuming the injectivity of $Ш^1(\mathbb{Q}, \mathsf{G}) \to Ш^1(\mathsf{F}, \mathsf{G})$, see Theorem 6.3.1.

6.1. **Igusa stacks.** Fix an isomorphism $\overline{\mathbb{Q}}_p \to \mathbb{C}$. For $L \subset \overline{\mathbb{Q}}_p$ let us denote by $\mathrm{ShTrp}_{L,\mathrm{Hdg}}$ the category whose objects are triples $(\mathsf{G}, \mathsf{X}, \mathcal{G})$ such that $\mathsf{E} \to \mathbb{C} \to \overline{\mathbb{Q}}_p$ factors through L and such that $(\mathsf{G}, \mathsf{X}, \mathcal{G})$ is of Hodge type, and whose morphisms are morphisms $f : (\mathsf{G}, \mathsf{X}) \to (\mathsf{G}', \mathsf{X}')$ such that f extends (necessarily uniquely) to a morphism $\mathcal{G} \to \mathcal{G}'$. By [5, Theorem I, Corollary 4.0.11], there is a functor $\mathscr{S}^{\diamond} : \mathrm{ShTrp}_{L,\mathrm{Hdg}} \to \mathcal{D}^{\circ}_{\mathcal{O}_L}$ sending $(\mathsf{G}, \mathsf{X}, \mathcal{G})$ to $\mathscr{S}_{K_p}(\mathsf{G}, \mathsf{X})^{\diamond}$, see Section 4.2.1. Here $K_p = \mathcal{G}(\mathbb{Z}_p)$ and $\mathscr{S}_{K_p}(\mathsf{G}, \mathsf{X})^{\diamond}$ is the diamond associated to the formal scheme $\widehat{\mathscr{S}}_{K_p}(\mathsf{G}, \mathsf{X})$, which is the p-adic completion of the integral model $\mathscr{S}_{K_p}(\mathsf{G}, \mathsf{X})$.

The following result is a consequence of [4, Theorem I, Theorem II] as we will explain below. It was originally conjectured by Scholze.

[8] In an earlier version of this paper, we proved a weaker version of these axioms for the same integral models of Shimura varieties in order to prove a version of Theorem 5.1.2. Our weaker version involved constructing shtukas for $\mathcal{G}^{\mathrm{ad}}$, which is easier than constructing $\mathcal{G}$-shtukas, see [6, Remark 1.1].

Theorem 6.1.1. *There is a functor* $\mathrm{Igs} : \mathrm{ShTrp}_{L,\mathrm{Hdg}} \to \mathcal{D}^{\circ}_{\mathcal{O}_L}$ *together with a natural transformation* $\mathscr{S}^{\diamond} \to \mathrm{Igs}$ *and a weak natural transformations* $\mathrm{Igs} \to \mathrm{Bun}$*, and a strictly commutative diagram of weak natural transformations*

$$\begin{array}{ccc} \mathscr{S}^{\diamond} & \longrightarrow & \mathrm{Sht} \\ \downarrow & & \downarrow \\ \mathrm{Igs} & \longrightarrow & \mathrm{Bun}, \end{array}$$

such that the resulting 2*-commutative diagrams of stacks on* Perf

$$\begin{array}{ccc} \mathscr{S}_{K_p}(\mathsf{G},\mathsf{X})^{\diamond} & \longrightarrow & \mathrm{Sht}_{\mathcal{G},\mu} \\ \downarrow & & \downarrow \\ \mathrm{Igs}_{K_p}(\mathsf{G},\mathsf{X}) & \longrightarrow & \mathrm{Bun}_G, \end{array}$$

are 2*-Cartesian for all* $(\mathsf{G},\mathsf{X},\mathcal{G}) \in \mathrm{ShTrp}_{L,\mathrm{Hdg}}$*, and such that* $\mathrm{Igs}_{K_p}(\mathsf{G},\mathsf{X}) \to \mathrm{Bun}_G$ *factors through the open substack* $\mathrm{Bun}_{G,\mu^{-1}} \to \mathrm{Bun}_G$*.*

Proof. The existence of $\mathrm{Igs}_{K_p}(\mathsf{G},\mathsf{X})$ is [4, Theorem II], the existence of the strict functor Igs and the strict natural transformation $\mathscr{S}^{\diamond} \to \mathrm{Igs}$ is [4, Theorem I]. The weak natural transformation $\mathrm{Igs} \to \mathrm{Bun}$ is constructed at the end of the proof of [4, Proposition 7.1.6], see the commutative cube there. By construction of the Igusa stack, the diagram

$$\begin{array}{ccc} \mathscr{S}_{K_p}(\mathsf{G},\mathsf{X})^{\diamond} & \longrightarrow & \mathrm{Sht}_{\mathcal{G},\mu} \\ \downarrow & & \downarrow \\ \mathrm{Igs}_{K_p}(\mathsf{G},\mathsf{X}) & \longrightarrow & \mathrm{Bun}_G, \end{array}$$

is strictly commutative for all $(\mathsf{G},\mathsf{X},\mathcal{G})$, and thus we see that the result follows. □

6.2. **Fixed points of Igusa stacks.** Let the notation be as in Section 5.1 and let $K_{p,1}^{\circ} \subset K_p \cap H_1(\mathbb{Q}_p)$ be the unique parahoric subgroup, corresponding to the identity component of $\mathcal{H}_1$.

Theorem 6.2.1. *If* $\text{Ш}^1(\mathbb{Q},\mathsf{G}) \to \text{Ш}^1(\mathsf{F},G)$ *is injective, then the natural map*

$$\mathrm{Igs}_{K_p^{\circ,\Gamma}}(\mathsf{G},\mathsf{X}) \to \mathrm{Igs}_{K_{p,1}^{\circ}}(\mathsf{H}_1,\mathsf{Y}_1)^{\Gamma} \times_{\mathrm{Bun}_{H_1}^{h\Gamma}} \mathrm{Bun}_{G,\mu^{-1}}$$

is an isomorphism.

To prove the theorem, we need to introduce potentially crystalline loci.

6.2.2. *The potentially crystalline locus in Hodge type Shimura varieties.* Recall that there is an open immersion $\mathbf{Sh}_K(\mathsf{G},\mathsf{X})^{\circ,\mathrm{an}} \subset \mathbf{Sh}_K(\mathsf{G},\mathsf{X})^{\mathrm{an}}$ of rigid spaces over E called the *potentially crystalline locus*, constructed in [13], see [13, Theorem 5.17]. The formation of $\mathbf{Sh}_K(\mathsf{G},\mathsf{X})^{\circ,\Diamond}$ is compatible with changing K, cf. [13, Corollary 5.29], and we will also consider

$$\mathbf{Sh}_{K_p}(\mathsf{G},\mathsf{X})^{\circ,\Diamond} = \varprojlim_{K^p \subset \mathsf{G}(\mathbb{A}_f^p)} \mathbf{Sh}_{K^pK_p}(\mathsf{G},\mathsf{X})^{\circ,\Diamond}$$

$$\mathbf{Sh}(\mathsf{G},\mathsf{X})^{\circ,\Diamond} = \varprojlim_{K \subset \mathsf{G}(\mathbb{A}_f^p)} \mathbf{Sh}_K(\mathsf{G},\mathsf{X})^{\circ,\Diamond}.$$

Lemma 6.2.3. *If $(\mathsf{G},\mathsf{X}) \to (\mathsf{G}',\mathsf{X}')$ is a closed immersion of Shimura data, then for $K_p' \subset G'(\mathbb{Q}_p)$ containing $K \subset G(\mathbb{Q}_p)$ there are equalities of open subdiamonds*

$$\mathbf{Sh}_{K_p}(\mathsf{G},\mathsf{X})^{\circ,\Diamond} = \mathbf{Sh}_{K_p'}(\mathsf{G}',\mathsf{X}')^{\circ,\Diamond} \times_{\mathbf{Sh}_{K_p'}(\mathsf{G}',\mathsf{X}')^{\Diamond}} \mathbf{Sh}_{K_p}(\mathsf{G},\mathsf{X})^{\Diamond}$$

$$\mathbf{Sh}(\mathsf{G},\mathsf{X})^{\circ,\Diamond} = \mathbf{Sh}(\mathsf{G}',\mathsf{X}')^{\circ,\Diamond} \times_{\mathbf{Sh}(\mathsf{G}',\mathsf{X}')^{\Diamond}} \mathbf{Sh}(\mathsf{G},\mathsf{X})^{\Diamond}.$$

Proof. This is a direct consequence of [13, Lemma 2.2], cf. [4, Lemma 5.1.6, Lemma 5.1.7]. □

The following lemma is [4, Lemma 5.1.6]

Lemma 6.2.4. *There is an equality*

$$\mathbf{Sh}_K(\mathsf{G},\mathsf{X})^{\circ,\Diamond} = \mathscr{S}_{K_p}(\mathsf{G},\mathsf{X})^{\diamond} \times_{\mathrm{Spd}\,\mathcal{O}_E} \mathrm{Spd}\,E$$

of open subdiamonds of $\mathbf{Sh}_K(\mathsf{G},\mathsf{X})^{\Diamond}$.

Lemma 6.2.5. *If $Ш^1(\mathbb{Q},\mathsf{G}) \to Ш^1(\mathsf{F},G)$ is injective, then the following diagram is 2-Cartesian*

$$\begin{array}{ccc} \mathbf{Sh}_{K_p^{\circ,\Gamma}}(\mathsf{G},\mathsf{X})^{\circ,\Diamond} & \longrightarrow & \mathbf{Sh}_{K_{p,1}^\circ}(\mathsf{H}_1,\mathsf{Y}_1)^{\circ,\Gamma,\Diamond} \\ \downarrow & & \downarrow \\ \mathrm{Sht}_{\mathcal{G},\mu,E} & \longrightarrow & \left(\mathrm{Sht}_{\mathcal{H}_1^\circ,\mu,E}\right)^{h\Gamma}. \end{array}$$

Proof. It follows from Lemma 3.6.7 that the natural map (where the limit runs over Γ-stable compact open subgroups of $\mathsf{H}(\mathbb{A}_f^p)$)

$$\varprojlim_{K \subset \mathsf{H}(\mathbb{A}_f^p)} \mathbf{Sh}_{K^\Gamma}(\mathsf{G},\mathsf{X})^{\Diamond} \to \varprojlim_{K \subset \mathsf{H}(\mathbb{A}_f^p)} \mathbf{Sh}_{K_1}(\mathsf{H}_1,\mathsf{Y}_1)^{\Diamond,\Gamma}$$

is an isomorphism. We then invoke Lemma 6.2.4 to get an isomorphism

$$\mathbf{Sh}(\mathsf{G},\mathsf{X})^{\circ,\Diamond} \to \mathbf{Sh}(\mathsf{H}_1,\mathsf{Y}_1)^{\circ,\Diamond}.$$

The map $\mathbf{Sh}(\mathsf{H}_1,\mathsf{Y}_1)^{\circ,\Diamond} \to \mathbf{Sh}_{K_{p,1}^\circ}(\mathsf{H}_1,\mathsf{Y}_1)^{\circ,\Diamond}$ is a torsor for the sheaf of groups $\underline{K_{p,1}^\circ}$ associated to the topological group $K_{p,1}^\circ$. Now we observe that there is a natural (in particular

Γ-equivariant) isomorphism

$$\mathrm{Sht}_{\mathcal{H}_1^\circ,\mu,E} \simeq \left[\mathrm{Gr}_{H,\mu^{-1}} / \underline{K_{p,1}^\circ}\right],$$

see [39, Proposition 11.17]. Moreover, by construction, the $\underline{K_{p,1}^\circ}$-torsor over $\mathbf{Sh}_{K_{p,1}^\circ}(\mathsf{H}_1,\mathsf{Y}_1)^{\circ,\lozenge}$ coming from the map

$$\mathbf{Sh}_{K_{p,1}}(\mathsf{H}_1,\mathsf{Y}_1)^{\circ,\lozenge} \to \mathrm{Sht}_{\mathcal{H}_1^\circ,\mu,E},$$

is given by $\mathbf{Sh}(\mathsf{H}_1,\mathsf{Y}_1)^{\circ,\lozenge} \to \mathbf{Sh}_{K_{p,1}^\circ}(\mathsf{H}_1,\mathsf{Y}_1)^{\circ,\lozenge}$. Thus we have a Γ-equivariant 2-Cartesian diagram

$$\begin{array}{ccc} \mathbf{Sh}(\mathsf{H}_1,\mathsf{Y}_1)^{\circ,\lozenge} & \longrightarrow & \mathrm{Gr}_{H,\mu^{-1}} \\ \downarrow & & \downarrow \\ \mathbf{Sh}(\mathsf{H}_1,\mathsf{Y}_1)^{\circ,\lozenge}/\underline{K_{p,1}^\circ} & \longrightarrow & \left[\mathrm{Gr}_{H,\mu^{-1}} / \underline{K_{p,1}^\circ}\right], \end{array}$$

whose homotopy fixed points are again 2-Cartesian by Lemma A.1.11. The homotopy fixed points diagram looks like (see the proof of Proposition 2.4.7)

$$\begin{array}{ccc} \mathbf{Sh}(\mathsf{G},\mathsf{X})^{\circ,\lozenge} & \longrightarrow & \mathrm{Gr}_{G,\mu^{-1}} \\ \downarrow & & \downarrow \\ \left(\mathbf{Sh}(\mathsf{H}_1,\mathsf{Y}_1)^{\circ,\lozenge}/\underline{K_{p,1}^\circ}\right)^{\Gamma} & \longrightarrow & \left[\mathrm{Gr}_{H,\mu^{-1}} / \underline{K_{p,1}^\circ}\right]^{h\Gamma}. \end{array}$$

If we base change the bottom row via $\mathbb{B}\underline{K_p^{\circ,\Gamma}} \to \mathbb{B}\underline{K_{p,1}^\circ}$, then by Proposition A.2.8 we get the Cartesian diagram

$$\begin{array}{ccc} \mathbf{Sh}(\mathsf{G},\mathsf{X})^{\circ,\lozenge} & \longrightarrow & \mathrm{Gr}_{G,\mu^{-1}} \\ \downarrow & & \downarrow \\ \mathbf{Sh}(\mathsf{G},\mathsf{X})^{\circ,\lozenge}/\underline{K_p^{\circ,\Gamma}} & \longrightarrow & \left[\mathrm{Gr}_{G,\mu^{-1}} / \underline{K_p^{\circ,\Gamma}}\right]. \end{array}$$

By Proposition A.2.8, this proves that the following diagram is Cartesian

$$\begin{array}{ccc} \mathbf{Sh}_{K_p^{\circ,\Gamma}}(\mathsf{G},\mathsf{X})^{\circ,\lozenge} & \longrightarrow & \mathbf{Sh}_{K_{p,1}^\circ}(\mathsf{H}_1,\mathsf{Y}_1)^{\circ,\Gamma,\lozenge} \\ \downarrow & & \downarrow \\ \left[\mathrm{Gr}_{G,\mu^{-1}} / \underline{K_p^{\circ,\Gamma}}\right] & \longrightarrow & \left[\mathrm{Gr}_{H,\mu^{-1}} / \underline{K_{p,1}^\circ}\right]^{h\Gamma}, \end{array}$$

proving the lemma. □

Proof of Theorem 6.2.1. It follows from Theorem 6.1.1 that there is a 2-commutative square

$$\begin{array}{ccc} \mathrm{Igs}_{K_p^\circ,\Gamma}(\mathsf{G},\mathsf{X}) & \longrightarrow & \mathrm{Bun}_{G,\mu^{-1}} \\ \downarrow & & \downarrow \\ \mathrm{Igs}_{K_{p,1}^\circ}(\mathsf{H}_1,\mathsf{Y}_1)^{h\Gamma} & \longrightarrow & \mathrm{Bun}_{H_1}^{h\Gamma}. \end{array}$$

This induces a map

$$\mathrm{Igs}_{K_p^\circ,\Gamma}(\mathsf{G},\mathsf{X}) \to \mathrm{Igs}_{K_{p,1}^\circ}(\mathsf{H}_1,\mathsf{Y}_1)^{h\Gamma} \times_{\mathrm{Bun}_{H_1}^{h\Gamma}} \mathrm{Bun}_{G,\mu^{-1}}$$

which we will show is an isomorphism. By v-descent we can do this after basechanging via the v-cover

$$\mathrm{Sht}_{\mathcal{G}^\circ,\mu,E} \to \mathrm{Bun}_{G,\mu^{-1}}$$

from [4, Corollary 6.4.2]. Using Theorem 6.1.1, Lemma 6.2.4 and Lemma A.1.11, we can identify the basechanged map with the natural map

$$\mathbf{Sh}_{K_p^\circ,\Gamma}(\mathsf{G},\mathsf{X})^{\circ,\lozenge} \to \mathbf{Sh}_{K_{p,1}^\circ}(\mathsf{H}_1,\mathsf{Y}_1)^{\circ,\lozenge,\Gamma} \times_{\mathrm{Sht}^{h\Gamma}_{\mathcal{H}_1^\circ,\mu,E}} \mathrm{Sht}_{\mathcal{G}^\circ,\mu,E}.$$

This natural map is an isomorphism by Lemma 6.2.5. □

6.3. **Fixed points of integral models of Shimura varieties of Hodge type II.** Let the notation be as in Section 6.2 above; the following result seems essentially optimal.

Theorem 6.3.1. *If $Ш^1(\mathbb{Q},\mathsf{G}) \to Ш^1(\mathsf{F},G)$ is injective, then the following diagram of v-stacks is 2-Cartesian*

$$\begin{array}{ccc} \mathscr{S}_{K_p^\circ,\Gamma}(\mathsf{G},\mathsf{X})^{\lozenge/} & \longrightarrow & \mathscr{S}_{K_{p,1}^\circ}(\mathsf{H}_1,\mathsf{Y}_1)^{\Gamma,\lozenge/} \\ \downarrow & & \downarrow \\ \mathrm{Sht}_{\mathcal{G}^\circ,\mu} & \longrightarrow & \mathrm{Sht}^{h\Gamma}_{\mathcal{H}_1^\circ,\mu}. \end{array}$$

Proof. If we basechange the isomorphism

$$\mathrm{Igs}_{K_p^\circ,\Gamma}(\mathsf{G},\mathsf{X}) \to \mathrm{Igs}_{K_{p,1}^\circ}(\mathsf{H}_1,\mathsf{Y}_1)^{h\Gamma} \times_{\mathrm{Bun}_{H_1}^{h\Gamma}} \mathrm{Bun}_{G,\mu^{-1}}$$

of Theorem 6.2.1 via $\mathrm{Sht}_{\mathcal{G}^\circ,\mu} \to \mathrm{Bun}_{G,\mu^{-1}}$, then by Theorem 6.1.1 we get the natural map

$$\mathscr{S}_{K_p^\circ,\Gamma}(\mathsf{G},\mathsf{X})^\diamond \to \mathscr{S}_{K_{1,p}^\circ}(\mathsf{H}_1,\mathsf{Y}_1)^{\Gamma,\diamond} \times_{\mathrm{Sht}^{h\Gamma}_{\mathcal{H}_1^\circ,\mu}} \mathrm{Sht}_{\mathcal{G}^\circ,\mu},$$

which is therefore an isomorphism of v-sheaves. If we combine this isomorphism with Lemma 6.2.5, then we see that the natural map

$$\mathscr{S}_{K_p^\circ,\Gamma}(\mathsf{G},\mathsf{X})^{\lozenge/} \to \mathscr{S}_{K_{p,1}^\circ}(\mathsf{H}_1,\mathsf{Y}_1)^{\Gamma,\lozenge/} \times_{\mathrm{Sht}^{h\Gamma}_{\mathcal{H}_1^\circ,\mu}} \mathrm{Sht}_{\mathcal{G}^\circ,\mu},$$

is an isomorphism (this follows from the definition of $\lozenge/$, see Section 2.3.3). This concludes the proof. □

Remark 6.3.2. If F is tamely ramified over $\mathbb{Q}$, then one can also use Theorem 3.6.4 to prove a version of Lemma 6.2.5 for well chosen $K^p \subset \mathsf{H}(\mathbb{A}_f^p)$. This will then lead to an analogue of Theorem 6.2.1 for the morphism of stacks[9]

$$\left[\mathrm{Igs}_{K_p^{\circ,\Gamma}}(\mathsf{G},\mathsf{X})/\underline{K^{p,\Gamma}}\right] \to \left[\mathrm{Igs}_{K'_{p,1}}(\mathsf{G},\mathsf{X})/\underline{K_1^p}\right]^{h\Gamma}$$

and to an analogue of Theorem 6.3.1 for the Shimura varieties of level $K^{\circ,\Gamma}$ and K'_1.

Appendix A. Some (2, 1)-category theory

A.1. **Strict (2, 1)-categories and weak functors.** Recall from [36, Section 02X8] the definition of a strict $(2,1)$-category. We will use $\mathrm{Mor}^1(x,y)$ to refer to the category of 1-morphisms between x and y in such a category, and sometimes abusively to refer to the class of objects of this category using the same notation. We will sometimes refer to isomorphisms in the category $\mathrm{Mor}^1(x,y)$ as natural transformations.

Definition A.1.1. Let $\mathcal{C}$ be a 1-category and $\mathcal{D}$ a strict $(2,1)$-category. We define a *weak functor* $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ to be a pair $(\mathcal{F}, \eta_{\mathcal{F}})$ consisting of:

(1) An assignment
$$\mathcal{F} : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$$
and a map $\mathcal{F} : \mathrm{Mor}(x,y) \to \mathrm{Ob}(\mathrm{Mor}^1(\mathcal{F}(x), \mathcal{F}(y)))$.

(2) For every $x \in \mathrm{Ob}(\mathcal{C})$ a 2-morphism $\eta_{\mathcal{F},x} : \mathrm{Id}_{\mathcal{F}(x)} \to \mathcal{F}(\mathrm{Id}_x)$.

(3) For every composable pair $f : x \to y$, $g : y \to z$ in $\mathcal{C}$ a 2-morphism $\eta_{\mathcal{F},f,g} : \mathcal{F}(g \circ f) \to \mathcal{F}(g) \circ \mathcal{F}(f)$.

Such that:

(1) For any morphism $f : x \to y$ in $\mathcal{C}$ we have that
$$\eta_{\mathcal{F},f,\mathrm{Id}_y} = \eta_{\mathcal{F},y} \circ \mathrm{Id}_{\mathcal{F}(f)}$$
and
$$\eta_{\mathcal{F},\mathrm{Id}_x,f} = \mathrm{Id}_{\mathcal{F}(f)} \circ \eta_{\mathcal{F},x}.$$

(2) For any composable triple $f : x \to y, g : y \to z, h : z \to t$ we have
$$(\mathrm{Id}_{\mathcal{F}(h)} \circ \eta_{\mathcal{F},f,g}) \circ \eta_{\mathcal{F},g\circ f,h} = (\eta_{\mathcal{F},g,h} \circ \mathrm{Id}_{\mathcal{F}(f)}) \circ \eta_{\mathcal{F},f,h\circ g}.$$

Definition A.1.2. A weak natural transformation $\epsilon : \mathcal{F} \to \mathcal{G}$ of weak functors $\mathcal{F}, \mathcal{G} : \mathcal{C} \to \mathcal{D}$ is:

(1) A collection for $x \in \mathrm{Ob}(\mathcal{C})$ of $\epsilon_x : \mathcal{F}(x) \to \mathcal{G}(x)$ of 1-morphisms in $\mathcal{D}$.

(2) For each morphism $f : x \to y$ in $\mathcal{C}$, a natural transformation $\epsilon(f) : \epsilon_y \circ \mathcal{F}(f) \to \mathcal{G}(f) \circ \epsilon_x$ in the category $\mathrm{Mor}^1(\mathcal{F}(x), \mathcal{G}(y))$, we will denote this in diagram form by

$$\begin{array}{ccc} \mathcal{F}(x) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(y) \\ \downarrow{\scriptstyle \epsilon_x} & \epsilon(f) & \downarrow{\scriptstyle \epsilon_y} \\ \mathcal{G}(x) & \xrightarrow{\mathcal{G}(f)} & \mathcal{G}(y), \end{array}$$

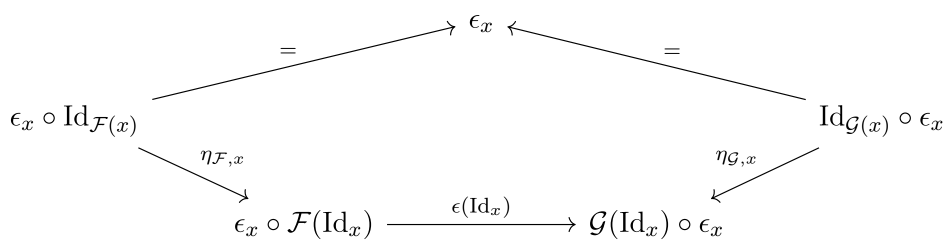
[9] We are now applying Theorem 6.1.1 with quasi-parahoric level, which is also proved in [4].

satisfying the following conditions:

(1) We have an equality of natural transformations for every pair of morphisms $f : x \to y, g : y \to z$:

$$\epsilon(g \circ f) \circ \eta_{\mathcal{F},f,g} = \eta_{\mathcal{G},f,g} \circ \mathcal{G}(g)_*(\epsilon(f)) \circ \mathcal{F}(f)^*(\epsilon(g)).$$

(2) For every object $x \in \mathrm{Ob}(\mathcal{C})$ we ask for commutativity of the diagram



inside of the category $\mathrm{Mor}(\mathcal{F}(x), \mathcal{G}(x))$. We note that because we work with strict $(2,1)$-categories the commutativity of this diagram is well-posed.

Let Γ be an abstract group. We define the category $B\Gamma$ to be the classifying category of the abstract group Γ (that is, the category with one object $*$ with automorphism group Γ). Suppose $\mathcal{D}$ is a strict $(2,1)$-category.

Definition A.1.3. A Γ-object in $\mathcal{D}$ is a weak functor $\mathcal{F} : B\Gamma \to \mathcal{D}$. A Γ-equivariant morphism is a weak natural transformation of such functors.

Let $\mathcal{D}$ be a strict $(2,1)$-category and let x be an object of $\mathcal{D}$. Then a weak functor $\mathcal{F} : B\Gamma \to \mathcal{D}$ sending $*$ to x is called a *weak* Γ*-action on* x.

Example A.1.4. Let $\mathcal{D}$ be a strict $(2,1)$-category and let x be an object of $\mathcal{D}$. Then the trivial Γ-action on x is the following weak functor $\mathcal{F} : B\Gamma \to \mathcal{D}$: On objects it sends $*$ to x and on morphisms it sends all morphisms to the identity $x \to x$. Moreover the 2-morphisms $\eta_{\mathcal{F},x}$ and $\eta_{\mathcal{F},f,g}$ are all taken to be the identity 2-morphism.

Example A.1.5. Let $\mathcal{D}$ be a strict $(2,1)$-category and let x be an object of $\mathcal{D}$ such that $\mathrm{Mor}^1(x,x)$ is a discrete category.[10] Then $\mathrm{Ob}(\mathrm{Mor}^1(x,x))$ is a group under composition of morphisms and a weak Γ-action on x is the same as a group homomorphism $\Gamma \to \mathrm{Ob}(\mathrm{Mor}^1(x,x))$.

A.1.6. Recall [36, Tag 003R] that fiber products in the $(2,1)$-category of categories have the following description. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be $(2,1)$-categories and let $\mathcal{F} : \mathcal{A} \to \mathcal{C}, \mathcal{G} : \mathcal{B} \to \mathcal{C}$ be functors. Then there is the following canonical presentation of the fiber product:

Definition A.1.7. The fiber product $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ identifies with the strict $(2,1)$-category whose objects consist of triples (a, b, f) where $a \in \mathrm{Ob}(\mathcal{A}), b \in \mathrm{Ob}(\mathcal{B})$ and $f : \mathcal{F}(a) \to \mathcal{G}(b)$ is an

[10] A discrete category is a category where the only morphisms are the identity morphisms.

isomorphism in $\mathcal{C}$. The morphisms $\phi : (a, b, f) \to (c, d, g)$ consist of pairs (X, Y) where $X : a \to c, Y : b \to d$ are such that the diagram

$$\begin{array}{ccc} \mathcal{F}(a) & \xrightarrow{f} & \mathcal{G}(b) \\ \downarrow{\scriptstyle X} & & \downarrow{\scriptstyle Y} \\ \mathcal{F}(c) & \xrightarrow{g} & \mathcal{G}(d) \end{array}$$

commutes.

Definition A.1.8. Let X be a Γ-object in the strict $(2, 1)$-category of categories, defined by some functor $\mathcal{F}$. Then we define the Γ-*homotopy fixed points* $X^{h\Gamma}$ of X to be the following category: The objects of $X^{h\Gamma}$ are tuples $(x, \{\tau_\gamma\}_{\gamma\in\Gamma})$, where $x \in \mathrm{Ob}(X)$ is an object and $\tau_\gamma : x \to \mathcal{F}(\gamma)(x)$ for each $\gamma \in \Gamma$ is an isomorphism such that for all γ, γ' the diagram

$$\begin{array}{ccc} x & \xrightarrow{\tau_{\gamma'}} & \mathcal{F}(\gamma')(x) \\ {\scriptstyle \tau_{\gamma'\gamma}}\downarrow & & \downarrow{\scriptstyle \mathcal{F}(\gamma')(\tau_\gamma)} \\ \mathcal{F}(\gamma'\gamma)(x) & \xrightarrow{\eta_{\mathcal{F},\gamma,\gamma'}} & \mathcal{F}(\gamma')(\mathcal{F}(\gamma)(x)) \end{array}$$

is commutative. A morphism $f : (x, \tau) \to (x', \tau')$ of such objects is a morphism $f : x \to x'$ such that the following diagram commutes for all $\gamma \in \Gamma$

$$\begin{array}{ccc} x & \xrightarrow{f} & x' \\ \downarrow{\scriptstyle \tau_\gamma} & & \downarrow{\scriptstyle \tau'_\gamma} \\ \mathcal{F}(\gamma)(x) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(\gamma)(x'). \end{array}$$

Lemma A.1.9. *A Γ-equivariant morphism $\alpha : X \to Y$ of Γ-objects in the strict $(2, 1)$-category of categories defines a natural functor*

$$\alpha^{h\Gamma} : X^{h\Gamma} \to Y^{h\Gamma}.$$

Furthermore, this functor is an equivalence if α is an equivalence.

Let $\mathcal{CAT}$ be the strict $(2, 1)$-category of categories and suppose that X and Y are given by weak functors $\mathcal{F}, \mathcal{G} : B\Gamma \to \mathcal{CAT}$. Then the morphism α is precisely a weak natural transformation $\epsilon : \mathcal{F} \to \mathcal{G}$. In particular, for each $\gamma \in \Gamma$ we have a natural transformation

$$\begin{array}{ccc} X & \xrightarrow{\mathcal{F}(\gamma)} & X \\ {\scriptstyle \alpha}\downarrow & \overset{\epsilon(f)}{\swarrow\!\!\!\swarrow} & \downarrow{\scriptstyle \alpha} \\ Y & \xrightarrow[\mathcal{G}(\gamma)]{} & Y. \end{array}$$

Given a tuple $(x, \{\tau_\gamma\}_{\gamma\in\Gamma}) \in X^{h\Gamma}$ we would like to define its image under $\alpha^{h\Gamma}$ to be the tuple $(\alpha(x), \{\epsilon_\gamma \circ \alpha(\tau_\gamma)\}_{\gamma\in\Gamma})$. To check that this tuple defines an object of $Y^{h\Gamma}$, we need to check that for $\gamma, \gamma' \in \Gamma$ the diagram

$$\begin{array}{ccc}
\alpha(x) & \xrightarrow{\epsilon_{\gamma'} \circ \alpha(\tau_{\gamma'})} & \mathcal{G}(\gamma')(\alpha(x)) \\
\Big\downarrow{\scriptstyle \epsilon_{\gamma'\gamma} \circ \alpha(\tau_{\gamma'\gamma})} & & \Big\downarrow{\scriptstyle \mathcal{G}(\gamma')(\epsilon_\gamma \circ \alpha(\tau_\gamma))} \\
\mathcal{G}(\gamma'\gamma)(\alpha(x)) & \xrightarrow{\eta_{\mathcal{G},\gamma,\gamma'}} & \mathcal{G}(\gamma')(\mathcal{G}(\gamma)(\alpha(x)))
\end{array}$$

commutes. But this is a direct consequence of the fact that ϵ is a weak natural transformation. Thus we can define $\alpha^{h\Gamma}$ on the level of objects by sending $(x, \{\tau_\gamma\}_{\gamma \in \Gamma}) \in X^{h\Gamma}$ to the tuple $(\alpha(x), \{\epsilon_\gamma \circ \alpha(\tau_\gamma)\}_{\gamma\in\Gamma}) \in Y^{h\Gamma}$.

Given a morphism $f : (x, \{\tau_\gamma\}_{\gamma\in\Gamma}) \to (x', \{\tau'_\gamma\}_{\gamma\in\Gamma})$ in $X^{h\Gamma}$, one can check that

$$\alpha(f) : \alpha(x) \to \alpha(x')$$

is a morphism in $Y^{h\Gamma}$. The association $f \mapsto \alpha(f)$ is compatible with compositions, since α is a functor from X to Y. Thus we have constructed the desired functor $\alpha^{h\Gamma}$.

If α is fully faithful, then it is immediate that $\alpha^{h\Gamma}$ is also fully faithful. If α is essentially surjective and fully faithful, then $\alpha^{h\Gamma}$ is also essentially surjective. Indeed, given $(y, \{\kappa_\gamma\}_{\gamma\in\Gamma}) \in Y^{h\Gamma}$ we may choose an object $x \in X$ and an isomorphism $\xi : \alpha(x) \to y$. For each $\gamma \in \Gamma$ there is an induced isomorphism

$$\alpha(\mathcal{F}(\gamma)) \to \mathcal{G}(\gamma)(y)$$

given by $\mathcal{G}(\gamma)(\xi) \circ \epsilon_\gamma$. Using the fully faithfulness, the isomorphism $\kappa_\gamma : y \to \mathcal{G}(\gamma)(y)$ induces a unique isomorphism $\tau_\gamma : x \to \alpha(x)$. Using this uniqueness, it is not hard to check that the tuple $(x, \{\tau_\gamma\}_{\gamma\in\Gamma})$ defines an object in $X^{h\Gamma}$. Its image under $\alpha^{h\Gamma}$ is moreover isomorphic to $(y, \{\kappa_\gamma\}_{\gamma\in\Gamma})$ by construction.

Lemma A.1.10. *Let $\mathcal{F}, \mathcal{G}, \mathcal{H} : \mathcal{C} \to \mathcal{CAT}$ where $\mathcal{C}$ is a 1-category and $\mathcal{CAT}$ is the strict $(2,1)$-category of categories. Let $\delta : \mathcal{F} \to \mathcal{H}, \epsilon : \mathcal{G} \to \mathcal{H}$ be weak natural transformations. Then there exists a canonical weak functor $\mathcal{F} \times_{\mathcal{H}} \mathcal{G} : \mathcal{C} \to \mathcal{CAT}$ such that $(\mathcal{F} \times_{\mathcal{H}} \mathcal{G})(x) = \mathcal{F}(x) \times_{\mathcal{H}(x)} \mathcal{G}(x)$ for all $x \in \mathrm{Ob}(\mathcal{C})$.*

Proof. **Step 1: Objects.** We define $\mathcal{P} := (\mathcal{F} \times_{\mathcal{H}} \mathcal{G})$ on objects by sending $x \mapsto \mathcal{F}(x) \times_{\mathcal{H}(x)} \mathcal{G}(x)$. For $x, y \in \mathcal{C}$ we define

$$\mathcal{P} : \mathrm{Mor}_{\mathcal{C}}(x, y) \to \mathrm{Ob}\left(\mathrm{Mor}(\mathcal{F}(x) \times_{\mathcal{H}(x)} \mathcal{G}(x), \mathcal{F}(y) \times_{\mathcal{H}(y)} \mathcal{G}(y))\right)$$

to be the assignment taking $g : x \to y$ to the functor

$$\mathcal{F}(g) \times_{\mathcal{H}(g)} \mathcal{G}(g) : \mathcal{F}(x) \times_{\mathcal{H}(x)} \mathcal{G}(x) \to \mathcal{F}(y) \times_{\mathcal{H}(y)} \mathcal{G}(y)$$

taking $(a, b, f) \in \mathcal{F}(x) \times_{\mathcal{H}(x)} \mathcal{G}(x)$ to the triple $(\mathcal{F}(g)(a), \mathcal{G}(g)(b), \mathcal{P}(g)(f))$, where $\mathcal{P}(g)(f))$ is defined as

$$\begin{array}{ccccc}
\delta_y(\mathcal{F}(g)(a)) & & \xrightarrow{\qquad\mathcal{P}(g)(f))\qquad} & & \\
\Big\downarrow{\scriptstyle \delta(g)} & & & & \\
\mathcal{H}(g)(\delta_x(a)) & \xrightarrow{\mathcal{H}(g)(f)} & \mathcal{H}(g)(\delta_x(b)) & \xrightarrow[\epsilon(g)^{-1}]{} & \epsilon_y(\mathcal{G}(g)(b)).
\end{array}$$

Step 2: Morphisms. A morphism $p : (a, b, f) \to (a', b', f')$ is sent to the morphism

$$(\mathcal{F}(g)(p), \mathcal{G}(g)(p))$$

which one checks is a morphism in $\mathcal{F}(y) \times_{\mathcal{H}(y)} \mathcal{G}(y)$. To check that $\mathcal{F}(g) \times_{\mathcal{H}(g)} \mathcal{G}(g)$ is a functor, one uses the fact that δ and ϵ are weak natural transformations (and thus have compatibility with composition).

Step 3: Identity natural transformations. Next, for each $x \in \mathrm{Ob}(\mathcal{C})$ we define a natural transformation $\eta_x : \mathrm{Id}_{\mathcal{P}(x)} \to (\mathcal{P})(\mathrm{Id}_x)$ given by the morphism

$$\eta_{x,(a,b,f)} : (a, b, f) \to (\mathcal{F}(\mathrm{Id}_x)(a), \mathcal{G}(\mathrm{Id}_x)(b), \mathcal{P}(\mathrm{Id}_x)(f))$$

described by $(\eta_{\mathcal{F},x,a}, \eta_{\mathcal{G},x,a})$. Naturality follows from the naturality of $\eta_{\mathcal{F}}$ and $\eta_{\mathcal{G}}$, and we must simply check that the morphisms $(\eta_{\mathcal{F},x,a}, \eta_{\mathcal{G},x,b})$ are morphisms in the category $\mathcal{F}(x) \times_{\mathcal{H}(x)} \mathcal{G}(x)$. We recall that this comes down to showing that the top square in the following diagram is commutative for every $(a, b, f) \in \mathrm{Ob}(\mathcal{F}(x) \times_{\mathcal{H}(x)} \mathcal{G}(x))$

(A.1.1)
$$\begin{array}{ccc}
\delta_x(a) & \xrightarrow{\quad f \quad} & \epsilon_x(b) \\
\Big\downarrow{\scriptstyle \delta_x(\eta_{\mathcal{F},x,a})} & & \Big\downarrow{\scriptstyle \epsilon_x(\eta_{\mathcal{G},x,b})} \\
\delta_x(\mathcal{F}(\mathrm{Id}_x)(a)) & \xrightarrow{\mathcal{P}(\mathrm{Id}_x)(f)} & \epsilon_x(\mathcal{G}(\mathrm{Id}_x)(b)) \\
\Big\downarrow{\scriptstyle \delta(\mathrm{Id}_x)} & & \Big\downarrow{\scriptstyle \epsilon(\mathrm{Id}_x)} \\
\mathcal{H}(\mathrm{Id}_x)(\delta_x(a)) & \xrightarrow{\mathcal{H}(\mathrm{Id}_x)(f)} & \mathcal{H}(\mathrm{Id}_x)(\epsilon_x(b)).
\end{array}$$

But we know that the bottom square is commutative by the definition of $\mathcal{P}(\mathrm{Id}_x)(f)$. Moreover, because δ is a weak natural transformation from $\mathcal{F}$ to $\mathcal{H}$, we know that that $\delta(\mathrm{Id}_x) \circ \delta_x(\eta_{\mathcal{F},x,a}) = \eta_{\mathcal{H},x} \circ \delta_x$ as morphisms between $\delta_x(a)$ and $\mathcal{H}(\mathrm{Id}_x)(\delta_x(a))$ in the category $\mathcal{H}(x)$. In addition $\epsilon(\mathrm{Id}_x) \circ \epsilon_x(\eta_{\mathcal{G},x,b}) = \eta_{\mathcal{H},x,b} \circ \epsilon_x$ as morphisms from $\epsilon_x(b)$ to $\mathcal{H}(\mathrm{Id}_x)(\epsilon_x(b))$. The commutativity of the outer square of (A.1.1) follows these last two observations. Since we also have commutativity of the bottom diagram, the commutativity of the top diagram follows since the vertical arrows are all isomorphisms.

Step 4: Natural transformations for compositions of morphisms. Finally, for a pair of morphisms $\gamma' : x \to y$ and $\gamma : y \to z \in \mathcal{C}$ we define a natural transformation $\eta_{\mathcal{P},\gamma',\gamma} : \mathcal{P}(\gamma' \circ \gamma) \to \mathcal{P}(\gamma') \circ \mathcal{P}(\gamma)$ of functors $\mathcal{P}(x) \to \mathcal{P}(z)$ which is defined on objects $(a, b, f) \in \mathcal{P}(x)$ by

$$(\eta_{\mathcal{F},\gamma,\gamma'}(a), \eta_{\mathcal{G},\gamma,\gamma'}(b)).$$

Once again, the naturality is straightforward if we can check that this is an isomorphism in $\mathcal{P}(z)$. For this, we consider the following diagram of isomorphisms in $\mathcal{H}(z)$

$$\begin{array}{ccc}
\mathcal{H}(\gamma' \circ \gamma)(\delta_x(a)) & \xrightarrow{\mathcal{H}(\gamma'\circ\gamma)(f)} & \mathcal{H}(\gamma' \circ \gamma)(\epsilon_x(a)) \\
{\scriptstyle \delta(\gamma'\gamma)}\big\uparrow & & \big\downarrow{\scriptstyle \epsilon(\gamma'\gamma)^{-1})} \\
\delta_z(\mathcal{F}(\gamma' \circ \gamma)(a)) & \xrightarrow{\mathcal{P}(\gamma'\circ\gamma)(f)} & \epsilon_z(\mathcal{G}(\gamma' \circ \gamma)(b)) \\
{\scriptstyle \delta_z(\eta_{\mathcal{F},\gamma,\gamma'}(a))}\big\downarrow & & \big\downarrow{\scriptstyle \epsilon_z(\eta_{\mathcal{G},\gamma,\gamma'}(b))} \\
\delta_z(\mathcal{F}(\gamma')(\mathcal{F}(\gamma)(a))) & \xrightarrow{\mathcal{P}(\gamma')(\mathcal{P}(\gamma)(f))} & \epsilon_z(\mathcal{G}(\gamma')(\mathcal{G}(\gamma)(b))) \\
\big\downarrow{\scriptstyle \delta(\gamma')} & & {\scriptstyle \epsilon(\gamma')^{-1}}\big\uparrow \\
\mathcal{H}(\gamma')(\delta_y(\mathcal{F}(\gamma)(a))) & \xrightarrow{\mathcal{H}(\gamma')(\mathcal{P}(\gamma)(f))} & \mathcal{H}(\gamma')(\epsilon_y(\mathcal{G}(\gamma)(a))) \\
\big\downarrow{\scriptstyle \mathcal{H}(\gamma')(\delta(\gamma))} & & {\scriptstyle (\mathcal{H}(\gamma')(\epsilon(\gamma))^{-1}}\big\uparrow \\
\mathcal{H}(\gamma')(\mathcal{H}(\gamma)(\delta_x(a))) & \xrightarrow{\mathcal{H}(\gamma')(\mathcal{H}(\gamma)(f))} & \mathcal{H}(\gamma')(\mathcal{H}(\gamma)(\epsilon_x(b))).
\end{array}$$

We are asked to show that the second square from the top commutes, and we see that this reduces to checking commutativity of the outer square in the diagram. But this again follows from the fact that δ and ϵ are weak natural transformations.

Step 5: End of the proof. We have now specified the data required to define a weak functor $\mathcal{C} \to \mathcal{CAT}$. To check that this is a weak functor, we need to check certain identities of natural transformations hold, see Definition A.1.1. Since all natural transformations $\eta_{\mathcal{P}}$ are defined as pairs $(\eta_{\mathcal{F}}, \eta_{\mathcal{G}})$, the identities for $\eta_{\mathcal{P}}$ follow from those for $\eta_{\mathcal{F}}$ and $\eta_{\mathcal{G}}$. □

We have the following key lemma. Let $\mathcal{F}, \mathcal{G}, \mathcal{H} : B\Gamma \to \mathcal{CAT}$ be weak functors and let $\delta : \mathcal{F} \to \mathcal{H}, \epsilon : \mathcal{G} \to \mathcal{H}$ be weak natural transformations. Recall the fiber product weak functor $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ from Lemma A.1.10.

Lemma A.1.11. *There is an equivalence of categories*

$$\beta : \mathcal{F}^{h\Gamma}(*) \times_{\mathcal{H}^{h\Gamma}(*)} \mathcal{G}^{h\Gamma}(*) \to (\mathcal{F} \times_{\mathcal{H}} \mathcal{G})(*)^{h\Gamma}.$$

Proof. Let us write $\mathcal{P} = \mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ for simplicity.

Step 1: Defining the functor on objects. The functor β has the following construction on objects: Let $(a, \{\tau_\gamma\}_{\gamma\in\Gamma})$ be an object in $\mathcal{F}^{h\Gamma}(*)$, let $(b, \{\sigma_\gamma\}_{\gamma\in\Gamma})$ be an object in $\mathcal{G}^{h\Gamma}(*)$ and let $f : \delta^{h\Gamma}(a, \{\tau_\gamma\}_{\gamma\in\Gamma}) \to \epsilon^{h\Gamma}(b, \{\sigma_\gamma\}_{\gamma\in\Gamma})$ be an isomorphism in $\mathcal{H}^{h\Gamma}(*)$. Observe that f is per definition an automorphism $\delta(a) \to \epsilon(b)$ in $\mathcal{H}$ satisfying certain extra properties.

We define $\beta((a, \{\tau_\gamma\}_{\gamma\in\Gamma}), (b, \{\sigma_\gamma\}_{\gamma\in\Gamma}), f)$ to be the triple $(a, b, f, \{\rho_\gamma\}_{\gamma\in\Gamma})$, where ρ_γ is the pair $(\tau_\gamma, \rho_\gamma)$. To check that our triple defines an element of $\mathcal{P}(*)^{h\Gamma}$, we need to check that for all γ, γ' the diagram

$$\begin{array}{ccc} (a,b,f) & \xrightarrow{\rho_\gamma} & \mathcal{P}(\gamma')(a,b,f) \\ \big\downarrow{\tau_{\gamma'\gamma}} & & \big\downarrow{\mathcal{P}(\gamma)(\rho_\gamma)} \\ \mathcal{P}(\gamma'\gamma)(a,b,f) & \xrightarrow{\eta_{\mathcal{P},\gamma,\gamma'}} & \mathcal{P}(\gamma')(\mathcal{P}(\gamma)(a,b,f)) \end{array} \tag{A.1.2}$$

commutes. But morphisms in $\mathcal{P}(*)^{h\Gamma}$ are determined by the corresponding morphisms in $\mathcal{P}(*)$, which are in turn determined by a pair consisting of a morphism in $\mathcal{F}(*)$ and a morphism in $\mathcal{G}(*)$. The commutativity of the diagram (A.1.2) then follows from the definition of $\eta_{\mathcal{P},\gamma,\gamma'}$ and $\mathcal{P}$, in combination with the fact that $(a, \{\tau_\gamma\}_{\gamma\in\Gamma})$ is an object in $\mathcal{F}^{h\Gamma}(*)$ and the fact that $(b, \{\sigma_\gamma\}_{\gamma\in\Gamma})$ is an object of $\mathcal{G}^{h\Gamma}(*)$.

Step 2: Defining the functor on morphisms. A morphism

$$g : ((a, \{\tau_\gamma\}_{\gamma\in\Gamma}), (b, \{\sigma_\gamma\}_{\gamma\in\Gamma}), f) \to ((a', \{\tau'_\gamma\}_{\gamma\in\Gamma}), (b', \{\sigma'_\gamma\}_{\gamma\in\Gamma}), f')$$

in $\mathcal{F}^{h\Gamma}(*) \times_{\mathcal{H}^{h\Gamma}(*)} \mathcal{G}^{h\Gamma}(*)$ corresponds to a pair of morphisms (g_1, g_2). Concretely, we have $g_1 : a \to a'$ and $g_2 : b \to b'$ such that the following diagrams commute for all $\gamma \in \Gamma$:

$$\begin{array}{ccc} a & \xrightarrow{g_1} & a' \\ \big\downarrow{\tau_\gamma} & & \big\downarrow{\tau'_\gamma} \\ \mathcal{F}(\gamma)(a) & \xrightarrow{\mathcal{F}(g_1)} & \mathcal{F}(\gamma)(a') \end{array} \quad \begin{array}{ccc} b & \xrightarrow{g_2} & b' \\ \big\downarrow{\sigma_\gamma} & & \big\downarrow{\sigma'_\gamma} \\ \mathcal{G}(\gamma)(b) & \xrightarrow{\mathcal{F}(g_2)} & \mathcal{G}(\gamma)(b') \end{array} \quad \begin{array}{ccc} \delta(a) & \xrightarrow{f} & \epsilon(b) \\ \big\downarrow{\delta(g_1)} & & \big\downarrow{\epsilon(g_2)} \\ \delta(a') & \xrightarrow{f'} & \epsilon(b'). \end{array} \tag{A.1.3}$$

We will define $\beta(g)$ to be the morphism

$$(a, b, f, \{\rho_\gamma\}_{\gamma\in\Gamma}) \to (a', b', f', \{\rho'_\gamma\}_{\gamma\in\Gamma})$$

corresponding to (g_1, g_2). This is a morphism in $\mathcal{P}(*)$ between (a, b, f) and (a', b', f) by the commutativity of the third diagram in (A.1.3). To show that this is a morphism in $\mathcal{P}(*)^{h\Gamma}$, we have to show that for all $\gamma \in \Gamma$ the diagram

$$\begin{array}{ccc} (a,b,f) & \xrightarrow{g} & (a',b',f') \\ \big\downarrow{\rho_\gamma} & & \big\downarrow{\rho'_\gamma} \\ P(\gamma)(a,b,f) & \xrightarrow{\mathcal{P}(\gamma)(g)} & P(\gamma)(a',b',f') \end{array} \tag{A.1.4}$$

commutes. But morphisms in $\mathcal{P}(*)$ are determined by pairs of morphisms in $\mathcal{F}(*)$ and $\mathcal{H}(*)$. The commutativity now follows from the commutativity of the first two diagrams of (A.1.3) and the definition of $P(\gamma)$ and ρ_γ.

Step 3: Showing β is fully faithful. Suppose we are given objects

$$((a, \{\tau_\gamma\}_{\gamma\in\Gamma}), (b, \{\sigma_\gamma\}_{\gamma\in\Gamma}), f), ((a', \{\tau'_\gamma\}_{\gamma\in\Gamma}), (b', \{\sigma'_\gamma\}_{\gamma\in\Gamma}), f') \in \mathcal{F}^{h\Gamma}(*) \times_{\mathcal{H}^{h\Gamma}(*)} \mathcal{G}^{h\Gamma}(*).$$

We have seen in Step 2 that a morphisms between these objects consists of a pair of morphisms $g_1 : a \to a'$ and $g_2 : b \to b'$ such that the diagrams in (A.1.3) commute. There are also conditions for the pair (g_1, g_2) to define a morphism between

$$\beta((a, \{\tau_\gamma\}_{\gamma\in\Gamma}), (b, \{\sigma_\gamma\}_{\gamma\in\Gamma}), f)), \beta((a', \{\tau'_\gamma\}_{\gamma\in\Gamma}), (b', \{\sigma'_\gamma\}_{\gamma\in\Gamma}), f')),$$

namely the commutativity of the third diagram in (A.1.3) and the commutativity of (A.1.4). But it is immediate that the latter condition is equivalent to the commutativity of the first two diagrams in (A.1.3), proving fully faithfulness.

Step 4: Essential surjectivity of β. Let $(a, b, f, \{\rho_\gamma\}_{\gamma\in\Gamma})$ be an object of $\mathcal{P}(*)^{h\Gamma}$. Then for each $\gamma \in \Gamma$ the map ρ_γ consists of a pair of an isomorphism $\tau_\gamma : a \to \mathcal{F}(\gamma)(a)$ and an isomorphism $\sigma_\gamma : b \to \mathcal{G}(\gamma)(b)$. It is straightforward to show that $(a, b, f, \{\rho_\gamma\}_{\gamma\in\Gamma})$ is equal to (!!) the image under β of the triple $(a, \{\tau_\gamma\}_{\gamma\in\Gamma}), (b, \{\sigma_\gamma\}_{\gamma\in\Gamma}), f)$. □

A.2. **Quotient stacks and fixed points.** In this section we will study the homotopy fixed points of quotients stacks and in particular the interaction between the formation of quotient stacks and homotopy fixed points.

Let S be a small category with all fiber products, equipped with a Grothendieck pretopology. We will consider the strict $(2,1)$-category $\mathrm{Cat}_{/S}$ of categories fibered in groupoids over S, see [36, Definition 02XS]. The 2-fiber product of categories fibered in groupoids as described in [36, Lemma 0040] is the same as the one described in Definition A.1.7, thus Lemma A.1.11 holds for categories fibered in groupoids.

Let G be a sheaf of groups on S and consider the category fibered in groupoids $\mathbb{B}G \to S$: Its objects are pairs $(\mathcal{P}, T)$ where T is an object of S and where $\mathcal{P} \to T$ is a left G-torsor. A morphism $(\mathcal{P}, T) \to (\mathcal{P}', T')$ is a pair of morphisms $a : T \to T'$ and $b : \mathcal{P} \to \mathcal{P}'$ such that b is G-equivariant and such that the following diagram commutes and is Cartesian

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{b} & \mathcal{P}' \\ \downarrow & & \downarrow \\ T & \xrightarrow{a} & T'. \end{array}$$

The composition of morphisms is given by concatenating Cartesian diagrams.

Recall the notation of a stack in groupoids over S, see [36, Definition 02ZI]. We observe that the proof of [36, Lemma 04UK] can be repeated to show that $\mathbb{B}G \to S$ is a stack in groupoids over S.

Given a morphism $f : G \to G'$ of sheaves of groups, there is a functor $\mathbb{B}f : \mathbb{B}G \to \mathbb{B}G'$ sending

$$(T, \mathcal{P}) \mapsto (T, \mathcal{P} \times^G G')$$

and on morphisms sending $b : \mathcal{P} \to \mathcal{P}'$ to the morphism $b' : \mathcal{P} \times^G G' \to \mathcal{P} \times^G G'$ defined by $b'(p, g') = (b(p), g')$. Here we consider $\mathcal{P} \times^G G'$ as the quotient (sheaf) of $P \times G'$ by the right action of G given by $(p, h) \cdot g = (g^{-1} \cdot p, hf(g))$. We let G' act on $P \times G'$ by $g' \cdot (p, h) = (p, g'h)$, which clearly descends to the quotient $\mathcal{P} \times^G G'$ turning that quotient into a left G'-torsor.

Lemma A.2.1. *Let Grp_S be the 1-category of sheaves of groups in S. There is a weak functor*

$$\mathbb{B} : \mathrm{Grp}_S \to \mathrm{Cat}_{/S}$$

which on objects sends G to $\mathbb{B}G$ and on morphisms sends $f : G \to G'$ to $\mathbb{B}f : \mathbb{B}G \to \mathbb{B}G'$.

Proof. Part (1) of Definition A.1.1 has been specified by the lemma. For part (2) we have to specify a 2-morphism

$$\eta_{\mathbb{B},G} : \mathrm{Id}_{\mathbb{B}G} \to \mathbb{B}(\mathrm{Id}_G).$$

We take the one which is given on objects by the map

$$\begin{aligned} \mathcal{P} &\to \mathcal{P} \times^G G \\ p &\mapsto (p, 1). \end{aligned}$$

Moreover, given $f_1 : G_1 \to G_2$ and $f_2 : G_2 \to G_3$ we have to specify a 2-morphism

$$\eta_{\mathbb{B},f_1,f_2} : \mathbb{B}(f_2 \circ f_1) \to \mathbb{B}(f_2) \circ \mathbb{B}(f_1).$$

We take the one given by the isomorphism

$$\begin{aligned} \mathcal{P} \times^{G_1} G_3 &\to \left(\mathcal{P}^{\times G_1} G_2\right) \times^{G_2} G_3 \\ (p, g_3) &\mapsto ((p, 1), g_3). \end{aligned}$$

It is straightforward, if somewhat tedious, to check that these coherence data satisfy Definition A.1.1 □

Now let X be a sheaf on S equipped with a left action of a sheaf of groups G. Then we define a category fibered in groupoids $[X/G] \to S$ whose objects are triples $(T, \mathcal{P}, \omega)$, where T is an object of S, where $\mathcal{P} \to T$ is a G-torsor and where $\omega : \mathcal{P} \to X$ is a G-equivariant map of sheaves. A morphism $(T, \mathcal{P}, \omega) \to (T', \mathcal{P}', \omega')$ is a pair (a, b), where $a : T \to T'$ is a morphism in S and $b : \mathcal{P} \to \mathcal{P}'$ is a G-equivariant morphism such that the following two diagrams commute

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{b} & \mathcal{P}' \\ \downarrow & & \downarrow \\ T & \xrightarrow{a} & T', \end{array} \qquad \begin{array}{ccc} \mathcal{P} & \xrightarrow{b} & \mathcal{P}' \\ \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \omega'} \\ X & \xrightarrow{\mathrm{Id}_X} & X, \end{array}$$

and such that the first diagram is Cartesian. We observe that $[X/G]$ is a stack in groupoids over S; indeed, the proof of [36, Lemma 0370] goes through verbatim.

Note that there is a natural forgetful morphism $[X/G] \to \mathbb{B}G$ of categories fibered in groupoids over S which sends $(T, \mathcal{P}, \omega) \mapsto (T, \mathcal{P})$.

A.2.2. Let GrpShv_S be the category whose objects are triples (G, X, A), where G is a sheaf of groups on S, where X is a sheaf on S, and where $A : G \times X \to X$ is a left action of G on X. A morphism $(G, X, A) \to (G', X', A')$ consists of a homomorphism of groups $h : G \to G'$ and a morphism of sheaves $f : X \to X'$, such that f is G-equivariant via h. There is an obvious forgetful functor $\mathrm{GrpShv}_S \to \mathrm{Grp}_S$ sending (G, X, A) to G. We consider $\mathbb{B}$ as a weak functor on GrpShv_S via this forgetful functor.

Lemma A.2.3. *There is a weak functor* $\mathscr{Q} : \mathrm{GrpShv}_S \to \mathrm{Cat}_{/S}$*, which on objects sends* (G, X, A) *to* $[X/G]$*. There is moreover a weak natural transformation* $\mathscr{Q} \to \mathbb{B}$*, which on objects induces the natural forgetful functor*

$$[X/G] \to \mathbb{B}G.$$

Proof. Given a morphism $(h, f) : (G, X, A) \to (G', X', A')$ we define a morphism

$$\mathscr{Q}(h, f) : [X/G] \to \left[X'/G'\right]$$

by sending $(T, \mathcal{P}, \omega) \mapsto (T, \mathcal{P}^{\times G} G', \omega_h)$, where

$$\begin{aligned} \omega_h : \mathcal{P}^{\times G} G' &\to X' \\ (p, g') &\mapsto A'(g', f(\omega(p)). \end{aligned}$$

For the identity natural transformation

$$\eta_{\mathscr{Q},(G,X,A)} : \mathrm{Id}_{\mathscr{Q}(G,X,A)} \to \mathscr{Q}(\mathrm{Id}_G, \mathrm{Id}_X)$$

we take the isomorphism $(T, \mathcal{P}, \omega) \mapsto (T, \mathcal{P}^{\times G} G, \omega_{\mathrm{Id}_G})$ given by

$$\begin{aligned} \mathcal{P} &\to \mathcal{P}^{\times G} G \\ p &\mapsto (p, 1), \end{aligned}$$

which is compatible with ω and ω_{Id_G} since

$$(\omega_{\mathrm{Id}_G})(p, 1) = A(1, \mathrm{Id}_X(\omega(p)) = \omega(p).$$

Given a pair of composable morphisms $(h, f) : (G_1, X_1, A_1) \to (G_2, X_2, A_2)$ and $(h', f') : (G_2, X_2, A_2) \to (G_3, X_3, A_3)$, we need to specify a natural transformation

$$\eta_{\mathscr{Q},(h,f),(h',f')} : \mathscr{Q}(h' \circ h, f' \circ f) \to \mathscr{Q}(h', f') \circ \mathscr{Q}(h, f).$$

For this we take the isomorphism in $[X_3/G_3]$ given by

$$(S, \mathcal{P} \times^{G_1} G_3, \omega_{h' \circ h}) \to (S, \left(\mathcal{P} \times^{G_2} G_2\right) \times^{G_2} G_3), (\omega_h)_{h'})$$

given by the isomorphism

$$\begin{aligned} \mathcal{P} \times^{G_1} G_3 &\to \left(\mathcal{P}^{\times G_1} G_2\right) \times^{G_2} G_3 \\ (p, g_3) &\mapsto ((p, 1), g_3). \end{aligned}$$

from the proof of Lemma A.2.1. We have to check that this is an isomorphism in $[X_3/G_3]$ which comes down to the equality

$$\omega_{h'h}(((p, 1), g_3)) = \omega_{h' \circ h}(p, g_3).$$

To check this we compute that

$$\begin{aligned} (\omega_h)_{h'}\left(((p, 1), g_3)\right) &= A'(g_3, f'(\omega_h(p, 1))) \\ &= A_3(g_3, f'(A_2(1, f(p)))) \\ &= A_3(g_3, f'(f(p))) \\ &= \omega_{h' \circ h}(p, g_3). \end{aligned}$$

As in the proof of Lemma A.2.1, we omit the verification that these coherence data satisfy Definition A.1.1.

The weak natural transformation $\mathscr{Q} \to \mathbb{B}$ is easy to specify because it follows by construction that for $(h, f) : (G, X, A) \to (G', X', A')$ the diagram

$$\begin{array}{ccc} [X/G] & \xrightarrow{\mathscr{Q}(h,f)} & [X'/G'] \\ \downarrow & & \downarrow \\ \mathbb{B}G & \xrightarrow{\mathbb{B}(h)} & \mathbb{B}G' \end{array}$$

is strictly commutative. Thus we can specify the identity natural transformation in (2) in Definition A.1.2, which clearly satisfies the properties outlined in that definition. □

We will now spell out a particular case of Lemma A.2.3: Let Γ be an abstract group, let $(G, X, A) \in \mathrm{GrpShv}_S$ and let $\Gamma \to \mathrm{Isom}_{\mathrm{GrpShv}_S}((G, X, A), (G, X, A))$ be a group homomorphism. This gives us a functor

$$B\Gamma \to \mathrm{GrpShv}_S,$$

which we can compose with the weak functor of Lemma A.2.3 to get a Γ-object in categories fibered in groupoids over S, see Example A.1.5. Concretely, the group homomorphism

$$\Gamma \to \mathrm{Isom}_{\mathrm{GrpShv}_S}((G, X, A), (G, X, A))$$

comes down to the following data: We have a left action of Γ on G over S; let us write $\alpha_\gamma : G \to G$ for the induced isomorphism of groups for each $\gamma \in \Gamma$. We have a left action of Γ on X, which we denote by $\beta_\gamma : X \to X$ for each $\gamma \in \Gamma$. These morphisms are subject to the following commutative diagram for all $\gamma \in \Gamma$

$$\begin{array}{ccc} G \times X & \xrightarrow{A} & X \\ \downarrow{\scriptstyle \alpha_\gamma \times \beta_\gamma} & & \downarrow{\scriptstyle \beta_\gamma} \\ G \times X & \xrightarrow{A} & X. \end{array}$$

The induced action of Γ on $[X/G]$ on objects takes a tuple $(T, \mathcal{P}, \omega)$ and sends it to the tuple $(T, \mathcal{P} \times^{G,\gamma} G, \omega_\gamma)$. Concretely, we have that

$$\omega_\gamma(p, g) = A(g, \beta_\gamma(\omega(p)))$$

which means that $\omega_\gamma(p, 1) = \beta_\gamma(\omega(p))$.

A.2.4. By Lemma A.2.3 there is a Γ-equivariant morphism

$$[X/G] \to \mathbb{B}G$$

and thus by Lemma A.1.9 there is a functor

$$[X/G]^{h\Gamma} \to (\mathbb{B}G)^{h\Gamma}.$$

Here $(\mathbb{B}G)^{h\Gamma}$ is the Γ-homotopy fixed points of $(\mathbb{B}G)$ in the sense of definition A.1.8, which is a category fibered in groupoids over S.

Lemma A.2.5. *The category fibered in groupoids* $(\mathbb{B}G)^{h\Gamma}$ *is a stack in groupoids over* S.

Proof. Descent for morphisms in $(\mathbb{B}G)^{h\Gamma}$ follows directly from descent of morphisms in $\mathbb{B}G$ since the forgetful map $(\mathbb{B}G)^{h\Gamma} \to \mathbb{B}G$ is faithful by construction. Descent for objects in $(\mathbb{B}G)^{h\Gamma}$ follows from descent for objects in $\mathbb{B}G$ in combination with descent for morphisms in $\mathbb{B}G$, and the weak functoriality of the Γ action. □

A.2.6. Now let G^Γ be the fixed point sheaf of G and let X^Γ be the fixed point sheaf of X. Then there is a natural morphism

$$\mathbb{B}G^\Gamma \to (\mathbb{B}G)^{h\Gamma}$$

of categories fibered in groupoids over S which sends $(T, \mathcal{P})$ to $(T, \mathcal{P} \times^{G^\Gamma} G)$ equipped with the isomorphisms

$$\tau_{\gamma,0} : \left(\mathcal{P} \times^{G^\Gamma} G\right) \to \left(\mathcal{P} \times^{G^\Gamma} G\right) \times^{G,\gamma} G$$
$$(p, g) \mapsto ((p, 1), \alpha_\gamma(g)),$$

and which sends a morphism

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{b} & \mathcal{P}' \\ \downarrow & & \downarrow \\ T & \xrightarrow{a} & T'. \end{array}$$

to the induced diagram

$$\begin{array}{ccc} \mathcal{P} \times^{G^\Gamma} G & \xrightarrow{b_G} & \mathcal{P}' \times^{G^\Gamma} G \\ \downarrow & & \downarrow \\ T & \xrightarrow{a} & T', \end{array}$$

where $b_G(p, g) = (b(p), g)$, which is well defined by G^Γ-equivariance of b. Similarly there is a natural morphism

$$\left[X^\Gamma/G^\Gamma\right] \to [X/G]^{h\Gamma}$$

of categories fibered in groupoids over S which sends $(T, \mathcal{P}, \omega)$ to $(T, \mathcal{P} \times^{G^\Gamma} G, \{\tau_{\gamma,0}\}_{\gamma\in\Gamma}, \omega_G)$, where

$$\omega_G : \mathcal{P} \times^{G^\Gamma} G \to X$$

is defined by $\omega_G(p, g) = A(g, \omega(p))$. By construction, we see that the following diagram of categories fibered in groupoids over S is strictly commutative

$$\begin{array}{ccc} \left[X^\Gamma/G^\Gamma\right] & \longrightarrow & [X/G]^{h\Gamma} \\ \downarrow & & \downarrow \\ (\mathbb{B}G^\Gamma) & \longrightarrow & (\mathbb{B}G)^{h\Gamma}. \end{array} \tag{A.2.1}$$

We let $\underline{H}^1(\Gamma, G)$ be the sheafification of the presheaf $T \mapsto H^1(\Gamma, G(T))$ on S. We have the following fundamental results.

Proposition A.2.7. *The morphism* $(\mathbb{B}G^\Gamma) \to (\mathbb{B}G)^{h\Gamma}$ *is an isomorphism if* $\underline{H}^1(\Gamma, G)$ *is trivial.*

Proposition A.2.8. *The diagram in equation* (A.2.1) *is* 2*-Cartesian.*

Corollary A.2.9. *If* $\underline{H}^1(\Gamma, G)$ *is trivial, then the natural map*

$$\left[X^\Gamma / G^\Gamma\right] \to [X/G]^{h\Gamma}$$

is an equivalence.

We start by proving a lemma.

Lemma A.2.10. *The natural map* $\mathbb{B}G^\Gamma \to (\mathbb{B}G)^{h\Gamma}$ *is fully faithful.*

Proof. Since this is a morphism of categories fibered in groupoids over S, it suffices to prove that the map on fibers is fully faithful. Let $T \in S$, let $\mathcal{P}, \mathcal{P}'$ be G^Γ-torsors over T and let $f : (T, \mathcal{P} \times^{G^\Gamma} G, \{\tau_{\gamma,0}\}_{\gamma \in \Gamma}) \to (T, \mathcal{P}' \times^{G^\Gamma} G, \{\tau'_{\gamma,0}\}_{\gamma \in \Gamma})$ be an isomorphism in $(\mathbb{B}G)^{h\Gamma}(T)$.

We need to show that f is induced from a unique isomorphism $\mathcal{P} \to \mathcal{P}'$. well defined up to left multiplication by $G^\Gamma(T)$. If we can show that $f_2(p, g) = g$, then f is induced from a unique isomorphism $\mathcal{P} \to \mathcal{P}'$; namely, the unique G^Γ-equivariant morphism sending p to $f_1(p, 1)$. It follows from the definitions of morphisms in $(\mathbb{B}G)^{h\Gamma}$ that for $\gamma \in \Gamma$ the following diagram must commute

$$\begin{array}{ccc} \left(\mathcal{P} \times^{G^\Gamma} G\right) & \xrightarrow{\tau_{\gamma,0}} & \left(\mathcal{P} \times^{G^\Gamma} G\right) \times^{G,\gamma} G \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle (f,1)} \\ \left(\mathcal{P}' \times^{G^\Gamma} G\right) & \xrightarrow{\tau'_{\gamma,0}} & \left(\mathcal{P}' \times^{G^\Gamma} G\right) \times^{G,\gamma} G. \end{array}$$

Using the definition $\tau_{\gamma,0}(p, g) = (p, 1, \gamma(g))$ this gives us the equality

$$((f_1(p, g), 1), \gamma(f_2(p, g)) = ((f_1(p, g), 1), \gamma(g)).$$

Thus we see that $f_2(p, g) = g$ and thus f is induced from the unique G^Γ-equivariant map $\mathcal{P} \to \mathcal{P}'$ sending $p \mapsto f_2(p, 1)$. □

A.2.11. Consider the sheaf $\underline{Z}^1(\Gamma, G)$ whose T-points are given by the set of cocycles $\underline{Z}^1(\Gamma, G(T))$ with values in $G(T)$. In other words, these are the functions $\sigma : \Gamma \to G(T)$ satisfying

$$\sigma(\gamma \cdot \gamma') = \sigma(\gamma) \cdot \alpha_\gamma(\sigma(\gamma')).$$

There is an action of G on $\underline{Z}^1(\Gamma, G)$, which takes an element $g \in G(T)$ and a cocycle $\sigma \in Z^1(\Gamma, G(T))$ and sends it to the cocycle

$$\sigma'(\gamma) = g \cdot \sigma(\gamma) \cdot \alpha_\gamma(g^{-1}).$$

The quotient set for this action is per definition the first (nonabelian) cohomology group of Γ with coefficients in $G(T)$, in formulas,

$$Z^1(\Gamma, G(T)/G(T) = H^1(\Gamma, G(T)).$$

Given a cocycle $\sigma : \Gamma \to G(T)$ and $\gamma \in \Gamma$ we get an isomorphism of trivial G_T-torsors

$$\begin{aligned} \epsilon_{\gamma,\sigma} : G_T &\to G_T \times^{G,\gamma} G \\ g &\mapsto (\alpha_{\gamma^{-1}}(g), \sigma(\gamma)). \end{aligned}$$

To check that $(T, G_T, \{\epsilon_{\gamma,\sigma}\}_{\gamma \in \Gamma}))$ defines an object of $(\mathbb{B}G)^{h\Gamma}$ we need to check that the following diagram commutes

$$\begin{array}{ccc} G_T & \xrightarrow{\epsilon_{\gamma_2}} & G_T \times^{G,\gamma_2} G \\ \Big\downarrow{\scriptstyle \epsilon_{\gamma_2\gamma_1,\sigma}} & & \Big\downarrow{\scriptstyle \mathbb{B}(\gamma_2)(\epsilon_{\gamma_1,\sigma})} \\ G_T \times^{G,\gamma_2\gamma_1} G & \xrightarrow{\eta_{\mathbb{B},\gamma_1,\gamma_2}} & \left(G_T \times^{G,\gamma_1} G\right) \times^{G,\gamma_2} G. \end{array}$$

This comes down to the equality

$$(\alpha_{\gamma_2^{-1}\gamma_1}(g), \sigma(\gamma_1\gamma_2)), 1) = ((\alpha_{\gamma_2^{-1}}\alpha_{\gamma_1^{-1}}(g), \sigma(\gamma_1)), \sigma(\gamma_1))$$

which follows from the cocycle condition

$$\sigma(\gamma \cdot \gamma') = \sigma(\gamma) \cdot \alpha_\gamma(\sigma(\gamma')).$$

If we now consider $\underline{Z}^1(\Gamma, G)$ as a fibered category over S, the construction above defines a natural map $\underline{Z}^1(\Gamma, G) \to (\mathbb{B}G)^{h\Gamma}$ which takes $(T, \sigma \in Z^1(\Gamma, G(T))$ to $(T, G_T, \{\epsilon_{\gamma,\sigma}\}_{\gamma \in \Gamma})$.

Lemma A.2.12. *This map induces an isomorphism*

$$\left[\underline{Z}^1(\Gamma, G)/G\right] \to (\mathbb{B}G)^{h\Gamma}.$$

Proof. This comes down to showing that the sheaf $\underline{Z}^1(\Gamma, G)$ represents the functor on S sending T to the set of Γ-equivariant structures on the trivial G-torsor over T. We have seen above how to go from a cocycle $\sigma \in \underline{Z}^1(\Gamma, G)(T)$ to a Γ-equivariant structure on the trivial G-torsor over T. Conversely, let $(\mathcal{P}^0 \times^{G^\sigma} G, \{\tau_\gamma\}_{\gamma \in \Gamma})$, be a Γ-equivariant structure on the trivial G-torsor $\mathcal{P}^0 \times^{G^\sigma} G$ where $\mathcal{P}^0$ is the trivial G^Γ-torsor. Then we get a cocycle by considering the collection of elements $\sigma(\gamma) \in G(T)$ defined to be the element of $G(T)$ corresponding to the following automorphism of $\mathcal{P}^0 \times^{G^\sigma} G$ over T

$$\mathcal{P}^0 \times^{G^\sigma} G \xrightarrow{\tau_\gamma} \left(\mathcal{P}^0 \times^{G^\sigma} G\right) \times^{G,\gamma} G \xrightarrow{\tau_{\gamma,0}^{-1}} \mathcal{P}^0 \times^{G^\sigma} G.$$

Here $\tau_{\gamma,0}$ is the isomorphism

$$\begin{aligned} \tau_{\gamma,0} : \left(\mathcal{P}^0 \times^{G^\Gamma} G\right) &\to \left(\mathcal{P}^0 \times^{G^\Gamma} G\right) \times^{G,\gamma} G \\ (p, g) &\mapsto ((p, 1), \gamma(g)) \end{aligned}$$

from the beginning of Section A.2.6. It is a straightforward yet laborious check that this defines an inverse to the construction above, and the lemma is proved. □

Proof of Proposition A.2.7. The functor is fully faithful by Lemma A.2.10. Essential surjectivity can be checked locally in the Grothendieck topology on S since both source and target are stacks in groupoids over S, see Lemma A.2.5. Thus given an object of $(\mathbb{B}G)^{h\Gamma}$ over an object T of S we may make a base change to assume that the underlying G-torsor is trivial. Then the Γ-equivariant structure corresponds to a cocycle $\sigma : \Gamma \to G(T)$, and the vanishing of $\underline{H}^1(\Gamma, G)$ tells us that we may make another base change to assume that this cocycle is trivial.

We now observe that the cocycle being trivial tells us precisely that the Γ-equivariant structure on the trivial G-torsor over T' is isomorphic to the *trivial* Γ-equivariant structure on the trivial G-torsor. In other words, that the corresponding object of $(\mathbb{B}G)^{h\Gamma}(T)$ is in the essential image of $(\mathbb{B}G^{h\Gamma})(T)$. $\square$

Proof of Proposition A.2.8. The fully faithfulness of the map follows from the fully faithfulness of $\mathbb{B}G^{\Gamma} \to (\mathbb{B}G)^{h\Gamma}$. Thus it suffices to prove essential surjectivity. The proposition then comes down to the following claim: Given an object $(T, \mathcal{P}, \{\tau_\gamma\}_{\gamma\in\Gamma}, \omega)$ of $[X/G]^{h\Gamma}$ such that there is a G^{Γ}-torsor $\mathcal{Q} \to T$ and a G^{Γ}-equivariant morphism $f : \mathcal{Q} \to \mathcal{P}$ inducing an isomorphism

$$\mathcal{P} \times^{G^{\Gamma}} G \to \mathcal{Q}$$

under which $\{\tau_\gamma\}_{\gamma\in\Gamma}$ corresponds to $\{\tau_{\gamma,0}\}_{\gamma\in\Gamma}$, the composition of f with ω factors through X^{Γ}. Let us write ω' for the map $\mathcal{Q} \times^{G^{\Gamma}} G \to X$ induced by f. In this situation the diagrams

$$\begin{array}{ccc} \mathcal{Q} \times^{G^{\Gamma}} G & \xrightarrow{\tau_{\gamma},0} & \left(\mathcal{Q} \times^{G^{\Gamma}}\right) \times^{G,\gamma} G \\ \downarrow{\scriptstyle \omega'} & & \downarrow{\scriptstyle \omega'_{\gamma}} \\ X & \xrightarrow{\mathrm{Id}_X} & X \end{array}$$

commute for all $\gamma \in \Gamma$, which means that for $q \in \mathcal{Q}$ we have

$$\omega'(q, 1) = \omega'_{\gamma}((q, 1), 1) = \beta_{\gamma}(\omega'(q, 1))$$

for all $\gamma \in \Gamma$. We deduce that ω' maps $\mathcal{Q} \subset \mathcal{Q} \times^{G^{\Gamma}} G$ to X^{Γ}, which concludes our proof of essential surjectivity. $\square$

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